9 Linear and constant-coefficient differential equations

9.1 Linear differential equations and the Wronskian

Here begins the great pivot of MATH 2400! Forget (not actually) everything we have previously discussed, as the rest of the course focused on differential equations rather than linear algebra.

Matrices do not go away, though, and the topics discusses in weeks 9-13 contain diff eq's with and without matrices and systems. So, think of it more as: weeks 9-10 are purely differential equations, and weeks 11-13 put everything from weeks 1-10 together.

What is a differential equation?

Simply an equation with at least one derivative in it. You've definitely encountered simple differential equations in other math classes and lower level calculus classes.
The most simple differential equation is really: $y' = ky$, where the derivative of y is equivalent to itself, multiplied by some constant, k.
Some important vocabulary about differential equations:

- Independent variable: variable/function that is changed, domain of function, most commonly t, time, or x, position
- Dependent variable: variable/function that is changed based on the value of the independent variable, most commonly x or y, position, such that $y(x)$ or $x(t)$ or $y(t)$. In differential equations, we differentiate with respect to the dependent variable(s)
- Linear: variables are combined linearly (addition/subtraction, we are not muliplying derivatives together)
- Homogeneous: differential equation that is set $= 0$
- Non-homogeneous: differential equation that is NOT set $= 0$
- Constant coefficient: equations where the coefficients of independent/dependent variables are **constants** (not functions)
- Non-constant coefficient: equations where the coefficients of independent/dependent variables are **not constants** (functions, independent variable, etc)
- Complementary/Homogeneous solution: y_c (sometimes y_h), the solution to a differential equation when it is homogeneous
- Particular solution: y_p, the solution associated with the non-homogeneous equation.
  - when a diff eq is homogeneous, $y_p = 0$
- General solution: $y = y_c + y_p$, the sum of the homogeneous and particular solutions - this is the "complete" solution to a diff eq
- Particular/specific solution: solved solution with specific constants (used in initial value problems)
- Initial value problems: differential equations with initial conditions that permit solving for specific constants

- First order: the highest derivatve in the equation is a first derivative, like y' or x'.
- Second order: the highest derivative in the equation is a second derivative, like y" or x". These equations don't *need* to have a first order derivative as well, but they can
  - The same follows for third order, fourth, etc.
- Separable: if a differential equation can have its independent and dependent variables separated (on different sides of the = sign) purely via algebraic manipulation
- Non-separable: cannot isolate independent and dependent variables with just algebraic manipulation, requires more methods to solve, rather than simple integration

Solving the most simple differential equation

Let's quickly solve the equation I mentioned earlier, $y' = ky$.

$y' = ky$ can be rewritten as: $\frac{dy}{dt} = ky$
Put like terms together (separate them): $\frac{1}{y}dy = kdt$
Integrate both sides: $\int \frac{1}{y}dy = \int kdt$
This gives us $\ln(y) = kt + \hat{C}$
And solving for y gives us $y = Ce^{kt}$ where $C = e^{\hat{C}}$

Recall that the original differential equation was $y' = ky$, where the derivative of y, a transformation of y, is *equivalent to multiplying y by a constant, k.*

Sound familiar? It should.

It's the same concept as eigenvalues and eigenvectors!

Here, we can say that the eigenvalue is k and the **eigenfunction** is e^{kt}. We will use this function a lot in the following sections.

Differential operator, D

Before we (temporarily) forget about matrices, the concept of a differential operator is important to discuss. It itself denotes differentiation on a function, vector, etc, such that $Df = f'$.

It acts as an operator and can be used to represent differential equations to replace the concept of a derivative. It is also a linear transformation. For example: $(D + 1)f = f' + f$

Further, we can represent the most "simple" differential equation, $y' = ky$, with the differential operator:

$$y' = ky$$
$$y' - ky = 0$$
$$Dy - ky = 0$$
$$(D - kI)y = 0$$

Quite similar to $(A - \lambda I)x = 0$, right?

Recalling solving this problem earlier, and these equivalencies, it is easy to see that the linear transformation, $D - kI$, is the one-dimensional subspace of functions with basis $\{e^{kt}\}$. Also, k is an eigenvalue of D with an eigenvector basis of $\{e^{kt}\}$.
There's a lot of details about this concept in the 9.1 notes, don't worry about it too much. It's there to show you where it came from, not what you'll be tested on.

Initial value problems

As defined in the definitions above, an initial value problem, IVP, is a differential equation with initial conditions that allow to solve for the constants in a solution.

In the context of our handy $y' = ky$ example, the **general solution** is $y = Ce^{kt}$.
To get the specific solution and make this an initial value problem, we need to be told an initial condition about the function, y, which could be something like $y(0) = 1$.

With this information, we can solve for C:

$$y = Ce^{kt} \to 1 = Ce^{k\times 0} \to 1 = Ce^0 \to 1 = C$$

The specific solution, is thus: $y = e^{kt}$

For a first order diff eq, you need one initial condition. For second order diff eq's, you need two initial conditions, and so on.
There is a theorem, the **Existence and uniqueness theorem**, which states that given a differential equation with necessary initial values, the equation will have a unique solution on any interval that contains the initial value and the functions are continuous.
Again, this derivation and proving it is not a MATH 2400 concept, but it's good to know, and with some subsequent derivations we arrive at...

the Wronskian

The Wronskian is a scalar quantity, calculated as:

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

Which basically means that the Wronskian of functions, in this case, $\{f_1, ...f_n\}$, is the **determinant** of those functions written columnwise with their derivatives in their column.
Here's a more specific example:

Wronskian of $(t^3, 3e^t)$

$$W(t^3, 3e^t) = \begin{vmatrix} t^3 & 3e^t \\ 3t^2 & 3e^t \end{vmatrix}$$

$$= t^3 \times 3e^t - 3e^t \times 3t^2$$
$$= 3t^3e^t - 9t^2e^t = 3t^2e^t(t-3)$$

Why do we care? Well, the Wronskian tells us a lot about a set of functions, especially in the context of differential equations:

- If $W \neq 0$ on some t interval, we know that those functions are **linearly independent** on that interval
- If $W = 0$ on some t interval, we know that those functions are **linearly dependent** on that interval

For MATH 2400, I want to add a few notes on these rules. It is a *very* common question on quizzes and finals to ask if functions are linearly independent or not, which means they want you to use the Wronskian. It is rare, however, for the question to give a *specific* t interval.
So, I find it beneficial to give more specific guidelines for answering these questions in MATH 2400 to make sure you get full credit.

If you are given a *specific interval* (like $t : [-2, 2]$)

- If $W = 0$ for ANY VALUE of $t \rightarrow$ **linearly dependent**
- If $W \neq 0$ for ALL VALUES of $t \rightarrow$ **linearly independent**

If you are NOT given a *specific interval*, assume the interval is $t : (-\infty, \infty)$

- If $W = 0$ for ALL VALUES of t ($W \equiv 0$) $\rightarrow$ **linearly dependent**
- If $W \neq 0$ for ANY VALUE of $t \rightarrow$ **linearly independent**

Continuing our discussion of the Wronskian, it doesn't seem too apparent why we are discussing this now, where are the diff eq's?
The Wronskian is a super relevant concept with solutions of differential equations because second order and beyond diff eq's have more than one unique solution. And, these solutions *must* be linearly independent.
Using the logic above, most questions on quizzes/finals is not just "Are these functions linearly independent?", it is more, "Can these functions be solutions to a constant coefficient linear differential equation?"
Which means we have to think a little bit more about what the Wronskian tells us:

$W \neq 0 \rightarrow$ linearly **independent** $\rightarrow$ CAN be solutions to a higher order diff eq
$W = 0 \rightarrow$ linearly **dependent** $\rightarrow$ CANNOT be solutions to a higher order diff eq

Let's discuss more on the Wronskian and second order diff eq's.

First, the standard form of a first order linear diff eq is:

$$y' + p(t)y = F(t)$$

- *Any* first order linear diff eq can be expressed in this form. You can always isolate y' in this manner through algebraic manipulation.
- If $p(t)$ is a constant ($\in \mathbb{R}$), the diff eq has **constant coefficients**.
- $p(t)$ has special qualities, as the coefficient of the second highest derivative in the diff eq (in this case, y^0), whether it is constant or a function. You *must* get the equation in standard form (isolate y') before you identify $p(t)$.
- If $F(t) = 0$, the diff eq is **homogeneous**; if $F(t) \neq 0$, the diff eq is **non-homogeneous**.
- First order diff eq's always have one unique solution.
- General solution: $\mathbf{y = Cy_1}$ (constant of integration times solution)

Now, the standard form of a second order linear diff eq is:

$$y'' + p(t)y' + q(t)y = F(t)$$

- *Any* second order linear diff eq can be expressed in this form. You can always isolate y'' in this manner through algebraic manipulation.
- If $p(t)$ is a constant ($\in \mathbb{R}$), the diff eq has **constant coefficients**.
- $p(t)$ has special qualities, as the coefficient of the second highest derivative in the diff eq (in this case, y'), whether it is constant or a function. You *must* get the equation in standard form (isolate y'') before you identify $p(t)$.
- If $F(t) = 0$, the diff eq is **homogeneous**; if $F(t) \neq 0$, the diff eq is **non-homogeneous**.
- Second order diff eq's always have two unique solutions.
- General solution: $\mathbf{y = c_1y_1 + c_2y_2}$ (constants of integration times each solution)
- We can take the Wronskian of the solutions of the diff eq, which will give us a function or a constant. We can also take the derivative of the Wronskian, W'.
  - Following some derivations (See MATH 2400 course notes 9.1.3 if you are interested, it's not too relevant), we arrive at the equation:
    $\mathbf{W' = -p(t)W}$,
    which relates W, W', and that important $p(t)$, the coefficient of y' in standard form
  - Further, if we solve for the solution to this first order diff eq, we get:
    $\mathbf{W = Ce^{\int p(t)}}$,
    which is also helpful, and we can see in following problems

The following are two different types of second order diff eq problems you are almost guaranteed to encounter on a quiz/exam. 9.2 focuses on how to directly solve second order diff eq's, but these are slightly different.

Given its two solutions, solve for the original linear homogeneous second order differential equation

$$y_1 = t \text{ and } y_2 = e^{3t}$$

Step 1: (Optional, unless asked to do) Determine linear independence

Before we get started with the question, we can confirm if these two functions can be solutions to a second order linear homogeneous equation using the Wronskian:

$$W(t, e^{3t}) = \begin{vmatrix} t & e^{3t} \\ 1 & 3e^{3t} \end{vmatrix}$$

$$= 3te^{3t} - e^{3t} = e^{3t}(3t - 1)$$

Remembering our rules from above, while this function *does* have a zero ($t = \frac{1}{3}$), it is *not* $\equiv 0$ (ALWAYS = 0)

So, they are **linearly independent**, and can be solutions to a second order diff eq

Let's recall that we are looking to generate a second order, linear, homogeneous differential equation, which takes the form: $y" + p(t)y' + q(t)y = 0$.

So, the things we have to find to define the diff eq are $p(t)$ and $q(t)$.

Step 2: Find $p(t)$

Luckily, we just got a way to relate the Wronskian to $p(t)$, and we already solved for W!

$$W' = -p(t)W$$
$$W = e^{3t}(3t-1) \rightarrow W' = 9te^{3t}$$

Plugging W and W' into $W' = -p(t)W$:

$$9te^{3t} = -p(t)e^{3t}(3t-1) \rightarrow 9t = -p(t)(3t-1)$$
$$\rightarrow p(t) = \frac{-9t}{3t-1}$$

And that's $p(t)$, so our diff eq is now: $y" - \frac{9t}{3t-1}y' + q(t)y = 0$

Step 3: Solve for $q(t)$

To fully define the equation, we lastly need $q(t)$. Luckily, we have everything else to plug into this partially solved equation, since we have $p(t)$ and y.
We have y, and thus y' and $y"$ because they were in the question! They are our solutions, so they will solve our differential equation.

We can either plug in $y_1 = t$ or $y_2 = e^{3t}$. To prove this, we can finish the problem either way. Normally, you only need to use one.

Solving for $q(t)$ with $y_1 = t$

$$y = t,\ y' = 1,\ y" = 0$$
$$y" - \frac{9t}{3t-1}y' + q(t)y = 0$$
$$(0) - \frac{9t}{3t-1}(1) + q(t)t = 0$$
$$\frac{9t}{3t-1} = q(t)t \rightarrow q(t) = \frac{9}{3t-1}$$

Solving for $q(t)$ with $y_1 = e^{3t}$

$$y = e^{3t},\ y' = 3e^{3t},\ y" = 9e^{3t}$$
$$y" - \frac{9t}{3t-1}y' + q(t)y = 0$$
$$(9e^{3t}) - \frac{9t}{3t-1}(3e^{3t}) + q(t)e^{3t} = 0$$
$$e^{3t}(9 - \frac{27t}{3t-1} + q(t)) = 0$$
$$\rightarrow 9 - \frac{27t}{3t-1} + q(t) = 0 \rightarrow q(t) = \frac{9}{3t-1}$$

No matter which solution we use, we get $q(t) = \frac{9}{3t-1}$.

Step 4: Form second order differential equation

Make sure to follow through with the question, which was to recover the original diff eq.
Putting it all together:

$$y" + p(t)y' + q(t)y = 0 \rightarrow y" + \frac{-9t}{3t-1}y' + \frac{9}{3t-1}y = 0$$

Which you could further simplify to: $(3t-1)y" - 9ty' + 9y = 0$. Both are correct.

Let's do the other type of problem, which is unfortunately much worse:

Given a second order differential equation and one solution, find the general solution

$y_1 = e^{\sqrt{t}}$ is one solution of $4ty" + (2 - 8\sqrt{t})y' + 3y = 0$
(gross, square roots, I know, but this was literally on a quiz)

Step 1: Get diff eq in standard form

Divide by the coefficient of y"

$$4ty" + (2 - 8\sqrt{t})y' + 3y = 0 \rightarrow y" + \frac{1-4\sqrt{t}}{2t}y' + \frac{3}{4t}y = 0$$

Step 2: Identify $p(t)$

$$p(t) = \frac{1-4\sqrt{t}}{2t}$$

Step 3: Relate $p(t)$ with W, W'

$$\text{Recall: } W' = -p(t)W$$

$$p(t) = \frac{1-4\sqrt{t}}{2t}$$

$$W' = -\frac{1-4\sqrt{t}}{2t}W$$

Step 4: Solve for W

This is a first order separable differential equation. We can solve this with algebraic manipulation and direct integration.

$$W' = -\frac{1-4\sqrt{t}}{2t}W$$

$$\frac{dW}{dt} = -\frac{1-4\sqrt{t}}{2t}W$$

$$\frac{1}{W}dW = -\frac{1-4\sqrt{t}}{2t}dt$$

$$\int \frac{1}{W}dW = \int -\frac{1-4\sqrt{t}}{2t}dt$$

NOTE: If you seeing the right hand side of this integral and thinking "I don't remember how to integrate that", this is the time to practice... unfortunately there are often some nasty integrals in this section of MATH 2400, there was even an inverse tangent integral on a final one semester. Reviewing integration in these weeks will benefit you a lot, especially if you are feeling a bit rusty.

$$\ln W = -\frac{1}{2}\ln t + 4\sqrt{t} \text{ (NOTE: not including } +C\text{)}$$

$$W = \frac{e^{4\sqrt{t}}}{\sqrt{t}}$$

Step 5: Solve for W, again

We can solve for W in a different manner. Remember, it is the determinant of the solutions of the diff eq with their respective derivatives. We don't have both solutions, but we *do* have one.

$$W(e^{\sqrt{t}}, y_2) = \begin{vmatrix} e^{\sqrt{t}} & y_2 \\ \frac{e^{\sqrt{t}}}{2\sqrt{t}} & y_2' \end{vmatrix} = e^{\sqrt{t}}y_2' - y_2\frac{e^{\sqrt{t}}}{2\sqrt{t}} = e^{\sqrt{t}}(y_2' - \frac{1}{2\sqrt{t}}y_2) = W$$

Step 6: Equate Ws

We have two different expressions for W, but we know they are equivalent.

$$W = \frac{e^{4\sqrt{t}}}{\sqrt{t}} = e^{\sqrt{t}}(y_2' - \frac{1}{2\sqrt{t}}y_2)$$

$$y_2' - \frac{y_2}{2\sqrt{t}} = \frac{e^{3\sqrt{t}}}{\sqrt{t}}$$

This leaves us with a first order differential equation. It is not separable, and we cannot solve this with algebraic manipulation and direct integration...

The integration factor

I wanted to introduce this method directly in this specific example, because the *only time* where you need to solve non-separable first order differential equations in this class, is going to be in this exact type of problem.

Remember, a first order differential equation is:

$$y' + p(t)y = F(t)$$

The integration factor is:

$$\mu(t) = e^{\int p(t)dt}$$

(I use μ to denote the integration factor, it doesn't really matter)
NOTE: This is kind of similar to $W = Ce^{\int p(t)}$ from the Wronskian section.
They are not the same.

The integration factor is used to convert a non-separable first order differential equation into an equation that can be directly integrated and separated.

Once a differential equation is in standard form (like above), we can multiply the entire equation by the integration factor, which will simplify the LHS (left hand side) into an easy integrable expression, because we are doing reverse derivative product rule.
The derivative product rule turns a product into two terms being added together. Here, we want to go the other way. The integration factor does this.

$$y' + p(t)y = q(t)$$

Multiply both sides by $\mu(t) = e^{\int p(t)dt}$:

$$e^{\int p(t)}y' + e^{\int p(t)}p(t)y = e^{\int p(t)}q(t)$$

LHS becomes: $\frac{d}{dt}(e^{\int p(t)}y)$ (ALWAYS the derivative of the product of $\mu(t)$ and y) so, equation is$\rightarrow$ $\frac{d}{dt}(e^{\int p(t)}y) = e^{\int p(t)}q(t)$

Don't believe it? Use the normal product rule on the LHS.

$\frac{d}{dt}(e^{\int p(t)}y) = e^{\int p(t)}y' + e^{\int p(t)}p(t)y \rightarrow$ This was the original LHS. It's a cool tool!

This makes the LHS, and also RHS, easily integrable, because fundamental theorem of calculus

$$\int \frac{d}{dt}(e^{\int p(t)}y) = \int e^{\int p(t)}q(t)$$

$$e^{\int p(t)}y = \int e^{\int p(t)}q(t)$$

Obviously, with actual expressions for $p(t)$ and $q(t)$ this will be fully solvable.
NOTE: The RHS here may contain some complicated integrals. You should review **integration by parts**.

After taking a quick integration factor instructions break, let's finish the problem.

Step 7: Solve for y_2

$$y_2' - \frac{y_2}{2\sqrt{t}} = \frac{e^{3\sqrt{t}}}{\sqrt{t}}$$

This equation is in standard form, so we can begin with the integration factor. Don't forget to get into standard form!

$$p(t) = -\frac{1}{2\sqrt{t}}$$

$$\mu(t) = e^{\int p(t)dt} = e^{-\frac{1}{2\sqrt{t}}dt} = e^{-\sqrt{t}}$$

Multiply first order diff eq by $\mu(t)$

$$e^{-\sqrt{t}}y_2' - e^{-\sqrt{t}}\frac{y_2}{2\sqrt{t}} = e^{-\sqrt{t}}\frac{e^{3\sqrt{t}}}{\sqrt{t}}$$

LHS becomes the derivative of the product of $\mu(t)$ and y_2; RHS requires algebraic simplification

$$\frac{d}{dt}(e^{-\sqrt{t}}y_2) = \frac{e^{2\sqrt{t}}}{\sqrt{t}}$$

Integrate both sides

$$\int \frac{d}{dt}(e^{-\sqrt{t}}y_2) = \int \frac{e^{2\sqrt{t}}}{\sqrt{t}}$$

$$e^{-\sqrt{t}}y_2 = \int \frac{e^{2\sqrt{t}}}{\sqrt{t}}$$

$$e^{-\sqrt{t}}y_2 = e^{2\sqrt{t}} \text{ (no } +C)$$

Solve for y_2

$$y_2 = e^{3\sqrt{t}}$$

And there is our second solution to the differential equation.

Step 8: Form general solution

Don't forget that the question asks for the general solution, *not* y_2. Don't do all of that calculus perfectly and lose points here.

$$y = c_1y_1 + c_2y_2 = c_1e^{\sqrt{t}} + c_2e^{3\sqrt{t}}$$

We neglected the constants of integration twice in this problem. They are here now, as constants to y_1 and y_2.

As you can tell, this type of problem is a lot of work. Practice it! It appears on exams a lot.

Now that we have learned about the world of differential equations, we can solve them in different manners in 9.2.