Week 4

Linear independence, basis and dimension

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4.1: Span, Linear Independence, and Basis

- Linear combinations (linear dependence)
- **Span**: how far a vector space can go
- **Linear independence (important for bases)**: set of vectors in V such that none of them can be written as a linear combination of each other except trivial solution
- **Basis**: set of vectors in V iff
  - Vectors are linearly independent
  - Vectors span the vector space
  - **Minimally spanning set**: remove one vector from set and no longer spans ***want this***

Linearly Independent Example

$$\begin{bmatrix}-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}-2\\3\\-2\end{bmatrix}$$

$$\begin{bmatrix}-3 & 1 & -2\\2 & -1 & 3\\-3 & 3 & -2\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Linearly Dependent Example

$$\begin{bmatrix}2\\-4\\5\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}, \begin{bmatrix}-1\\-6\\9\end{bmatrix}$$

$$\begin{bmatrix}2 & 3 & -1\\-4 & 2 & -6\\5 & -4 & 9\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

4.1: Standard Bases

- Every vector space has infinitely many bases, but there are some standard bases to recognize
- Every minimally spanning basis for a vector space has the same number of vectors in the basis
  - A basis CANNOT have linearly dependent components
- **Dimension**: common number of vectors in the vector space V
  - Dim (P_d) = d+1
  - Dim (R^n) = n

Some standard bases:

$\mathbb{R}^2$ is the space of all vectors in the x, y plane, and its standard basis is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathbb{R}^3$ is the space of all vectors in the x, y, z plane, and its standard basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

P_3 is the space of all polynomials (functions) up to the degree of 3. its standard basis is $\{1, x, x^2, x^3\}$

$\mathbb{R}^{2x2}$ is the space of all 2×2 matrices, and its standard basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

4.1.1: Linear independence example

Determine if the following sets of 3×1 vectors are linearly independent:

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 9 \end{bmatrix} \right\}$$

4.1 Worksheet questions

1. *Review.* Let A be an $n \times n$ matrix with $\det A = 3$. What is $\det(-2A)$?

2. *Review: recognizing vector spaces.* Is the following set S a vector space? $S =$ the set of rank one 3×3 matrices together with the zero matrix.

3. *Linear independence and bases.* Let V be the space of homogeneous quadratic polynomials in two variables, i.e. polynomials of the form $f(x, y) = ax^2 + bxy + cy^2$.

a) Are elements x^2, xy, y^2 linearly independent?

b) What about x^2, $x^2 + xy + y^2$, $xy + y^2$?

c) And x^2, $x^2 + xy$, $x^2 + xy + y^2$?

d) What is the dimension of this vector space V?

4.1 Worksheet questions

4. *Subspaces.* Let V be the vector space of homogeneous quadratic polynomials in two variables from the previous problem. Consider the subset of V that consists of functions $f \in V$ such that $f(1,1) = 0$. Denote this subset by W – is it a vector subspace of V?

a) Prove that W either is or is not a vector subspace of V. If W is a subspace, then complete parts b) and c):

b) Find a basis of W.

c) What is the dimension of W?

4.2: Change of Basis

- There are infinitely many bases for any given vector space, and every basis for a given space has the same dimension and span

- Every vector in a vector space is represented by the coefficients used to generate the vector using its basis elements, traditionally we use the standard basis

- When changing between bases, we can multiply a vector that is currently with respect to one basis by a **change of basis (COB) matrix** to change it to another basis: $P_{S \to B} v_S = v_B$

- Taking the inverse of a COB matrix changes the direction of the change: $P_{S \to B} = (P_{B \to S})^{-1}$

- We can "squish" together COB matrices to move between multiple bases: $P_{S \to B} P_{B \to C} = P_{S \to C}$

4.2: Rank and Nullity

- **Row space**
  - vector space generated by rows of a matrix (rows in RREF)
  - dimension of the row space: **rank**
- **Column space**
  - vector space generated by columns of a matrix (original columns from RREF)
  - dimension of the column space: **rank**
- **Nullspace**
  - aka kernel
  - vector space generated by all vectors such that $Ax=0$
  - solve $Ax=0$ as described in week 2
  - dimension of nullspace: **nullity**

4.2: Rank Nullity Theorem

$$\mathbf{Rank}(\mathbf{A}) + \mathbf{Nullity}(\mathbf{A}) = \mathbf{n}$$

The number of pivot variables in the RREF of a matrix corresponds to the rank of that matrix, and then the number of free variables in the RREF of a matrix corresponds to the nullity of that matrix.

4.2.1: Change of Basis Example 1

Let $B = b_1, b_2 = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $B' = b'_1, b'_2 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^2$, where each is relative to the standard basis $S = s_1, s_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Find $P_{B' \leftarrow B}$.

4.2.2: Change of Basis Example 2

Let $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^2$. You are told the matrix representation of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis B_1 on the source and B_2 on the target, is given by ${}_{B_2}M_{B_1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

What is ${}_{C_2}M_{C_1}$, where $C_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $C_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$?

Suppose B_3 is another basis of $\mathbb{R}^2$, where ${}_{B_3}M_{B_1} = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}$. What is the change of basis matrix $P_{B_3 \leftarrow B_2}$?

4.2.2: Change of Basis Example 3

Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of $\mathbb{R}^2$ and $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ be another basis.

If $\begin{bmatrix} 5 \\ 0 \end{bmatrix}_S = \begin{bmatrix} a \\ b \end{bmatrix}_B$, what are a and b?

If $\begin{bmatrix} 5 \\ 0 \end{bmatrix}_B = \begin{bmatrix} c \\ d \end{bmatrix}_S$, what are c and d?

4.2.3: Rank/Nullity theorem Example

Find the basis for the row space, column space, and nullspace of the following matrix:

$$\begin{bmatrix} 3 & 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 5 & 6 \\ 2 & 4 & 3 & 7 & 8 \end{bmatrix}$$

4.2 Worksheet questions

1. *Change of bases.* Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree $\leqslant 3$. Let $\mathcal{W}$ be the subspace of $\mathcal{P}_3$ consisting of polynomials that satisfy $p(1) = 0$ and $p(-1) = 0$. One basis of this subspace is $\mathcal{B}_1 = \{1 - x^2, x - x^3\}$. Another basis is $\mathcal{B}_2 = \{1 - x - x^2 + x^3, 2 + x - 2x^2 - x^3\}$

(a) What polynomial is denoted $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}_1}$?

(b) What polynomial is denoted $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}_2}$?

(c) Express the polynomials in the basis $\mathcal{B}_2$ in terms of the basis $\mathcal{B}_1$

(d) What is the change of basis matrix $P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1}$?

2. *Rank/nullity theorem.* For the following matrices, find the dimensions and bases for the row space, column space and nullspace:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 0 \\ 3 & 1 & 1 \\ 4 & 4 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}.$$