Week 2

Gaussian Elimination, Systems of Equations, Matrix Inverses

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2.1: Systems of Equations

- We can represent systems of linear equations as matrices and solve using gaussian elimination
- These equations form an equation of the form $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A}$ is an mxn matrix, $\boldsymbol{x}$ is a nx1 solution vector, and $\boldsymbol{b}$ comes from the constants in the system
    - Setting $\boldsymbol{b} = \boldsymbol{0}$ allows us to solve for the homogeneous system, $\boldsymbol{Ax} = \boldsymbol{0}$

solutions of $Ax = b$	
No solution	Inconsistent
One solution	Consistent
Infinitely many solutions	

$$\begin{aligned} 2x + 3y &= 5 \\ -5x - 4y &= 8 \end{aligned} \rightarrow \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

2.1: Gaussian Elimination

- **Gaussian elimination**: operations to the rows (or columns) to simplify the system to find its solution
- **Echelon form**: first entry in each row is a leading 1
- **Reduced row echelon form (RREF)**: leading 1 are the only non-zero entries in their respective columns (**what we want**)
- **Kernel/nullspace**: set of vectors x such that $Ax = 0$
- **Pivot variable**: variables that's column has a leading 1 in RREF
- **Free variable**: variable whose value is dependent on pivot variables
- **Rank**: # non-zero rows in RREF, also the # of pivot variables
  - Tells us a LOT about the system — if rank = # columns, unique solutions

Types of row operations	
Type 1	Switching two rows
Type 2	Scaling a row
Type 3	Adding a scaled row to another row

2.1.1: RREF Example

$$\left[\begin{array}{ccc|c} 5 & 10 & 5 & 15 \\ 3 & 12 & -9 & 9 \\ 4 & -4 & 12 & 8 \end{array}\right]$$

2.1.2: Finding Solution Set Example

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2.1.3 Defined solutions example

Consider the system of linear equations

$$\left\{\begin{array}{c} x+y+z=b_1 \\ 2x+y+z=b_2 \\ 3x+2y+2z=b_3 \end{array}\right\}$$

(a) Are there any conditions on b_1, b_2, b_3 that would result in the system having *no solutions*? If so, provide such conditions. If not, explain why the system must have at least one solution for any $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(b) Are there any conditions on b_1, b_2, b_3 that would result in the system having *infinitely many solutions*? If so, provide such conditions. If not, explain why the system cannot have infinitely many solutions for any $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(c) Are there any conditions on b_1, b_2, b_3 that would result in the system having a *unique* solution? If so, provide such conditions. If not, explain why the system cannot have a unique solution for any $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

2.1 Worksheet questions

1. *In this problem, we will practice block multiplication.* Consider the following column vector c and a 3×3 matrix A with columns a_1, a_2, a_3:

$$c = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}, \; A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}.$$

   a) Write the result of matrix multiplication Ac as a linear combination of the column vectors a_1, a_2, a_3.

   b) What if we write a matrix R as three rows $R = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{pmatrix}$ and multiply R by A?

2.1 Worksheet questions

2. *Gaussian elimination.* Consider the following system of linear equations:

$$\begin{aligned} 2x - y + 2z &= -5, \\ x - 3y + z &= 5, \\ x - 2y + z &= 2. \end{aligned}$$

Let v denote the vector of variables:

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

a) Find a 3×3 matrix A and a 3-vector b so that the above system can be written as $Av = b$.
b) Write the augmented matrix $[A \mid b]$ and perform Gaussian elimination on it.
c) What is the rank of A? What is the rank of $[A \mid b]$?
d) Describe $\operatorname{Ker} A$.
e) How many solutions does this system have? Why? If it does, find all of them.

2.2: Inverse of a square matrix

- In order to solve our system, $Ax = b$, we can isolate x by multiplying both sides of the equation by the **inverse**, A^{-1}, such that $x = A^{-1}b$
- **Invertible**: the inverse of the matrix exists iff (if and only if)
  - It is square (nxn)
  - Rank = n
- $AA^{-1} = A^{-1}A = I$
- Computing the inverse:
  - Augment A with I
  - Row reduce until A is the identity
  - I is now A^{-1}

- Invertible = nonsingular
- Non-invertible = singular

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{T})^{-1} = (A^{-1})^{T}$

For a 2x2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.2.1: Matrix Inverse Example

$$\begin{bmatrix} 5 & 10 & 5 \\ 3 & 12 & -9 \\ 4 & -4 & 12 \end{bmatrix}$$

2.2 Worksheet questions

1. Let A be a square $n \times n$ matrix, and let B and C be other $n \times n$ matrices such that $BA = I = AC$. Prove that then $B = C$.

2. In this problem, we will find the 2×2 matrix A that satisfies the following matrix equalities:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \text{ and } A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

   a) Combine the two given equalities into one: $AB = C$, where B and C are two 2×2 matrices.
   b) Use the formula from the lecture to compute B^{-1}.
   c) Multiply both sides of $AB = C$ by B^{-1} on the right and perform matrix multiplication to find A.