4 Linear independence, basis and dimension

4.1 Linear independence, bases and dimension

Determine if the following sets of 3×1 vectors are linearly independent:

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\}$$

Does the system

$$c_1 \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 & -2 \\ 2 & -1 & 3 \\ -3 & 3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -3 & 1 & -2 & 0 \\ 2 & -1 & 3 & 0 \\ -3 & 3 & -2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

have a non-trivial solution? ($c_1 \neq 0, c_2 \neq 0$, or $c_3 \neq 0$)
↳ linearly dependent
↳ if only trivial solution
⇒ linearly independent

↳ only solution:
$c_1 = 0$
$c_2 = 0$ (trivial solution) ⇒ $\boxed{\text{linearly independent}}$
$c_3 = 0$

$$\left\{ \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 9 \end{bmatrix} \right\}$$

Does the system

$$c_1 \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & -1 \\ -4 & 2 & -6 \\ 5 & -4 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ -4 & 2 & -6 & 0 \\ 5 & -4 & 9 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

have a non-trivial solution? ($c_1 \neq 0, c_2 \neq 0$, or $c_3 \neq 0$)
↳ linearly dependent
↳ if only trivial solution
⇒ linearly independent

↳ $\left.\begin{array}{l} c_1 + c_3 = 0 \\ c_2 - c_3 = 0 \end{array}\right\}$ This also has the trivial solution $c_1 = c_2 = c_3 = 0$
$0 + 0 = 0$, $0 - 0 = 0$,
but it also has non-trivial solutions!
∴ $\boxed{\text{linearly dependent}}$
↳ any constants $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} s$, $s \in \mathbb{R}$ will solve the system!

4.2 Change of basis, the rank/nullity theorem

Let $B = \{b_1, b_2\} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and $B' = \{b'_1, b'_2\} = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$ be two bases for $\mathbb{R}^2$, where each is relative to the standard basis $S = \{s_1, s_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Find $P_{B' \leftarrow B}$.

we know $(P_{S \leftarrow B'})(P_{B' \leftarrow B}) = P_{S \leftarrow B}$

$$P_{B' \leftarrow B} = (P_{S \leftarrow B'})^{-1}(P_{S \leftarrow B}) = (P_{B' \leftarrow S})(P_{S \leftarrow B}) \leadsto P_{B' \leftarrow B} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -3 & -4 \end{bmatrix}$$

$P_{S \leftarrow B}$ $P_{S \leftarrow B'}$ $(P_{S \leftarrow B'})^{-1} = P_{B' \leftarrow S}$

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \;\&\; \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1}$$

Let $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ be two bases for $\mathbb{R}^2$. You are told the matrix representation of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis B_1 on the source and B_2 on the target, is given by ${}_{B_2}M_{B_1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

What is ${}_{C_2}M_{C_1}$, where $C_1 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $C_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$?

$P_{S\leftarrow B_1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$P_{S\leftarrow B_2} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$P_{S\leftarrow C_1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

$P_{S\leftarrow C_2} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$
\begin{aligned}
{}_{C_2}M_{C_1} &= P_{C_2\leftarrow S} \cdot P_{S\leftarrow B_2} \cdot {}_{B_2}M_{B_1} \cdot P_{B_1\leftarrow S} \cdot P_{S\leftarrow C_1} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = {}_{C_2}M_{C_1}
\end{aligned}
$$

ALSO:

$$
\begin{aligned}
{}_{C_2}M_{C_1} &= P_{C_2\leftarrow B_2} \cdot {}_{B_2}M_{B_1} \cdot P_{B_1\leftarrow C_1} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = {}_{C_2}M_{C_1}
\end{aligned}
$$

Suppose B_3 is another basis of $\mathbb{R}^2$, where ${}_{B_3}M_{B_1} = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}$. What is the change of basis matrix $P_{B_3\leftarrow B_2}$?

$${}_{B_3}M_{B_1} = P_{B_3\leftarrow B_2} \cdot {}_{B_2}M_{B_1}$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = P_{B_3\leftarrow B_2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

$$\boxed{P_{B_3\leftarrow B_2} = \begin{bmatrix} -\frac{17}{2} & \frac{9}{2} \\ -\frac{9}{2} & \frac{5}{2} \end{bmatrix}}$$

Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of $\mathbb{R}^2$ and $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ be another basis.

If $\begin{bmatrix} 5 \\ 0 \end{bmatrix}_S = \begin{bmatrix} a \\ b \end{bmatrix}_B$, what are a and b?

$$P_{S \leftarrow B} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

* want to go from S to B, need $P_{B \leftarrow S}$ *

$$P_{B \leftarrow S} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{2(2)-1(3)} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_B = P_{B \leftarrow S} \begin{bmatrix} 5 \\ 0 \end{bmatrix}_S = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix}_S = \boxed{\begin{bmatrix} 10 \\ -5 \end{bmatrix}_B}$$

If $\begin{bmatrix} 5 \\ 0 \end{bmatrix}_B = \begin{bmatrix} c \\ d \end{bmatrix}_S$, what are c and d?

$$P_{S \leftarrow B} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

* want to go from S to B, need $P_{S \leftarrow B}$ *

$$\begin{bmatrix} c \\ d \end{bmatrix}_S = P_{S \leftarrow B} \begin{bmatrix} 5 \\ 0 \end{bmatrix}_B$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix}_B = \boxed{\begin{bmatrix} 10 \\ 5 \end{bmatrix}_S}$$

Find the basis for the row space, column space, and nullspace of the following matrix:

$$\begin{bmatrix} 3 & 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 5 & 6 \\ 2 & 4 & 3 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 5 & 6 \\ 2 & 4 & 3 & 7 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 4/3 & 7/4 \\ 0 & 0 & 1 & 1/3 & -2/3 \end{bmatrix} \longrightarrow$$

Nullspace:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 4/3 & 7/4 & 0 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + \frac{1}{3}x_4 + \frac{1}{3}x_5 = 0 \to x_1 = \frac{-1}{3}x_4 - \frac{1}{3}x_5 \\ x_2 + \frac{4}{3}x_4 + \frac{7}{3}x_5 = 0 \to x_2 = \frac{-4}{3}x_4 - \frac{7}{3}x_5 \\ x_3 + \frac{1}{3}x_4 - \frac{2}{3}x_5 = 0 \to x_3 = \frac{-1}{3}x_4 + \frac{2}{3}x_5 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1/3 \\ -7/3 \\ 2/3 \\ 0 \\ 1 \end{bmatrix} x_5$$

$\hookrightarrow$ Nullsp $= \left\{ \begin{bmatrix} -1/3 \\ -4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ -7/3 \\ 2/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Column space:

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(columns with pivot variables in RREF, but original columns)

Row space:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 4/3 & 7/3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1/3 & -2/3 \end{bmatrix} \right\}$$

(rows with pivot variables in RREF, rows from RREF!)