MATH 2400 Spring 2024 Final

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April 2025

Math 2400 Spring 2024 Final Exam

Question 1

SET-UP the matrix form of the system of equations that is used to find the equation of the parabola $y = ax^2 + bx + c$ that goes through the points $(1, 5), (-1, 3)$ and $(3, -1)$.
DO NOT SOLVE.

Question 2

Find a basis for the subspace of polynomials spanned by

$$x^3 + 3x^2 - 5x, \quad -2x^3 + x^2, \quad 2x^2 + x - 1, \quad x^3 - 4x^2 + 5x.$$

Question 3

Let

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

(a) Find the column space of A.

(b) Find the nullspace of A.

(c) Find all eigenvalues of A. Is A diagonalizable? Justify your answer.

Question 4

Suppose that V is the plane in R^3 given by the equation

$$x - 2y + 2z = 0.$$

Consider the bases $\mathcal{A} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$ of the plane V.

(a) Find the change of basis matrix $P_{\mathcal{A} \leftarrow \mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{A}$.

(b) If the $\mathcal{A}$-coordinate vector of $\mathbf{v}$ is $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$, what is the $\mathcal{B}$-coordinate vector of $\mathbf{v}$?

Question 5

(a) Determine the 4×4 matrix P which orthogonally projects a vector in R^4 onto the subspace V spanned by

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

(b) What is

$$P^{10} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

where P is given in (a)?

Question 6

Find the general solution $y(t)$ of the differential equation

$$y'' - 2y' + y = 2e^t.$$

Question 7

(a) $y_1(t) = t$ is a solution of the differential equation

$$ty'' - y' + \frac{y}{t} = 0.$$

Find the general solution of the differential equation.

(b) Find a second-order homogeneous linear differential equation with constant coefficients, such that $y_1(t) = e^t \cos(t)$, and $y_2(t) = e^t \sin(t)$ are two solutions.

Question 8

Find the general solution of the differential equation below using variation of parameters:

$$y'' - 10y' + 25y = \frac{2e^{5t}}{1+t^2}$$

Question 9

Determine the general solution $\mathbf{x}(t)$ to the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \mathbf{x}.$$

Question 10

Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}$$

1. Of course, the origin is the only critical point of this system. What kind of critical point is it?
2. Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

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Question 1

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DO NOT SOLVE.

$y = ax^2 + bx + c$

$(1,5) \Rightarrow 5 = a(1)^2 + b(1) + c$

$5 = a + b + c$

$(-1,3) \Rightarrow 3 = a(-1)^2 + b(-1) + c$

$3 = a - b + c$

$(3,-1) \Rightarrow -1 = a(3)^2 + b(3) + c$

$-1 = 9a + 3b + c$

$AX = y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

Question 2

Find a basis for the subspace of polynomials spanned by

$$x^3 + 3x^2 - 5x, \quad -2x^3 + x^2, \quad 2x^2 + x - 1, \quad x^3 - 4x^2 + 5x.$$

$$\begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ -5 & 0 & 1 & 5 \\ 3 & 1 & 2 & -4 \\ 1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ pivot variables correspond to basis elements (linearly ind.)

$\Rightarrow$ $\boxed{\{x^3+3x^2-5x,\ -2x^3+x^2,\ 2x^2+x-1\}}$

Question 3

Let

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

(a) Find the column space of A.

(b) Find the nullspace of A.

(c) Find all eigenvalues of A. Is A diagonalizable? Justify your answer.

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 1
Nullity = 4

$x_1 = -x_2 - x_3 - x_4 - x_5$

$$\text{Colsp}(A) = \left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$$

$$\text{Nullsp}(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

eigenvalues:

$\sum \text{rows} = \sum \text{column} = 15 \;\therefore\; \lambda = 15$

$\text{tr}(A) = 15 \rightarrow \sum \lambda = 15$

$\text{Nullity}(A) = 4 \rightarrow \text{Nullity}(A - 0I) = 4 \rightarrow \chi(\lambda) = (\lambda - 15)\lambda^4$

A is diagonalizable

1. symmetric matrix
2. alg. = geo.

$\lambda = 15$	$\lambda = 0$
alg. = geom = 1	alg. = geom. = 4

Question 4

Suppose that V is the plane in R^3 given by the equation

$$x - 2y + 2z = 0.$$

Consider the bases $\mathcal{A} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$ of the plane V.

(a) Find the change of basis matrix $P_{\mathcal{A}\leftarrow\mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{A}$.

(b) If the $\mathcal{A}$-coordinate vector of $\mathbf{v}$ is $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$, what is the $\mathcal{B}$-coordinate vector of $\mathbf{v}$?

$P_{A\leftarrow B}$: Find the A coordinate vectors of the basis B

$$\begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}_B = a_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 0 \\ 2 \end{bmatrix}_A$$

$$\begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}_B = a_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -2 & 6 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}_A$$

$$P_{A\leftarrow B} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$$

A coordinate vector : $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$, Find B coordinate vector

USE $P_{B\leftarrow A}$

$$P_{A\leftarrow B} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} \quad \therefore \quad P_{B\leftarrow A} = \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$V_B = P_{B\leftarrow A} \cdot V_A$$

$$= \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$V_B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}_B$$

SOLVE BY HAND

$$V_A = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$V = 8 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 20 \\ 8 \\ -2 \end{bmatrix} = b_1 \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -4 & 6 & 20 \\ 0 & 2 & 8 \\ 2 & -1 & -2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix}_B = V_B$$

Question 5

(a) Determine the 4×4 matrix P which orthogonally projects a vector in R^4 onto the subspace V spanned by

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

(b) What is

$$P^{10} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

where P is given in (a)?

Looking for matrix P that transforms the same as an orthogonal projection,

so $x \in \mathbb{R}^4 \Rightarrow Px = (x \cdot e_v)e_v$, where e_v is the unit vector of v

$$e_v = \frac{1}{\sqrt{1+4+1+4}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$$

columns of P are standard basis

$$Pb_1 = (b_1 \cdot e_v)e_v = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 1/10 \\ 2/10 \\ 1/10 \\ 2/10 \end{bmatrix}$$

$$Pb_2 = (b_2 \cdot e_v)e_v = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 2/10 \\ 4/10 \\ 2/10 \\ 4/10 \end{bmatrix}$$

$$Pb_3 = (b_3 \cdot e_v)e_v = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 1/10 \\ 2/10 \\ 1/10 \\ 2/10 \end{bmatrix}$$

$$Pb_4 = (b_4 \cdot e_v)e_v = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 2/10 \\ 4/10 \\ 2/10 \\ 4/10 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/10 & 2/10 & 1/10 & 2/10 \\ 2/10 & 4/10 & 2/10 & 4/10 \\ 1/10 & 2/10 & 1/10 & 2/10 \\ 2/10 & 4/10 & 2/10 & 4/10 \end{bmatrix}$$

$P^{10}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$? since P is a projection $P^n(x) = P(x)$ for all $x \in \mathbb{R}^4$

$$\longrightarrow P^{10}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = P\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \\ 3/5 \\ 6/5 \end{bmatrix} = P^{10}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Question 6

Find the general solution $y(t)$ of the differential equation

$$y'' - 2y' + y = 2e^t.$$

1. solve homogeneous

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = 1$, alg. mult = 2

$y_c = c_1 e^t + c_2 t e^t$

2. Solve particular

$F(t) = 2e^t$

$y_c = c_1 e^t + c_2 t e^t$ } overlap!

guess must = At^2e^t

$q = At^2e^t$ $\quad q' = At^2e^t + 2Ate^t$ $\quad q'' = At^2e^t + 2Ate^t + 2Ate^t + 2Ae^t$

$y'' - 2y' + y = 2e^t$

$At^2e^t + 2Ate^t + 2Ate^t + 2Ae^t - 2At^2e^t - 4Ate^t + At^2e^t = 2e^t$

$At^2e^t - 2At^2e^t + At^2e^t + 2Ate^t + 2Ate^t - 4Ate^t + 2Ae^t = 2e^t$

$2Ae^t = 2e^t$

$\therefore A = 1$

$y_p = t^2e^t$

$y = y_c + y_p$

$y = c_1e^t + c_2te^t + t^2e^t$

Question 7

(a) $y_1(t) = t$ is a solution of the differential equation

$$ty'' - y' + \frac{y}{t} = 0.$$

Find the general solution of the differential equation.

(b) Find a second-order homogeneous linear differential equation with constant coefficients, such that $y_1(t) = e^t \cos(t)$, and $y_2(t) = e^t \sin(t)$ are two solutions.

$ty'' - y' + \frac{y}{t} = 0$

S.F.

$y'' - \frac{1}{t}y' + \frac{y}{t^2} = 0$

$p(t) = \frac{-1}{t}$

$W' = -pW \Rightarrow \frac{dW}{dt} = \frac{1}{t}W \Rightarrow \int \frac{1}{W}dW = \int \frac{1}{t}dt$

$e^{\ln(W)} = e^{\ln(t)}$

$W = t$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & y_2 \\ 1 & y_2' \end{vmatrix} = ty_2' - y_2 = W$

$ty_2' - y_2 = t$

S.F.

$y_2' - \frac{1}{t}y_2 = 1$

$\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\int \frac{1}{t}dt} = \frac{1}{e^{\int \frac{1}{t}dt}} = \frac{1}{e^{\ln(t)}} = \frac{1}{t}$

$\frac{1}{t}\left[y_2' - \frac{1}{t}y_2 = 1\right]$

$\int \frac{d}{dt}\left[y_2 \frac{1}{t}\right] = \int \frac{1}{t}$

$\frac{y_2}{t} = \ln(t)$

$y_2 = t\ln(t) \longrightarrow$ $\boxed{y = c_1 t + c_2 t\ln(t)}$

$y_1 = e^t\cos(t) \quad y_2 = e^t\sin(t)$

SOLVE WITH WRONSKIAN

$W = \begin{vmatrix} e^t\cos(t) & e^t\sin(t) \\ -e^t\sin(t) + e^t\cos(t) & e^t\cos(t) + e^t\sin(t) \end{vmatrix}$

$= e^{2t}\cos^2(t) + e^{2t}\sin(t)\cos(t) - \left[-e^{2t}\sin^2(t) + e^{2t}\cos(t)\sin(t)\right]$

$= e^{2t}\cos^2(t) + e^{2t}\sin^2(t) + \sout{e^{2t}\sin(t)\cos(t) - e^{2t}\cos(t)\sin(t)}$

$= e^{2t}(1) = e^{2t} = W, \quad W' = 2e^{2t}$

$W' = -pW$

$2e^{2t} = -p(t)e^{2t}$

$p(t) = -2$

$y'' - 2y' + q(t)y = 0 \Rightarrow -2e^t\sin(t) - 2(-e^t\sin(t) + e^t\cos(t)) + q(t)(e^t\cos(t)) = 0$

$= -2e^t\sin(t) + 2e^t\sin(t) - 2e^t\cos(t) + q(t)(e^t\cos(t)) = 0$

$y = e^t\cos(t)$

$y' = -e^t\sin(t) + e^t\cos(t)$

$y'' = -e^t\cos(t) + -e^t\sin(t) + -e^t\sin(t) + e^t\cos(t)$

$= -2e^t\sin(t)$

$2e^t\cos(t) = q(t)e^t\cos(t)$

$2 = q(t)$

$\boxed{y'' - 2y' + 2y = 0}$

SOLVE VISUALLY

$y = c_1e^t\cos(t) + c_2e^t\sin(t)$

$r = a \pm bi = 1 \pm i$

$= (r - (1+i))(r - (1-i)) = 0$

$= r^2 - (1+i)r - (1-i)r + (1+i)(1-i) = 0$

$= r^2 - r - ir - r + ir + 1 - i^2 = 0$

$= r^2 - 2r + 1 - (-1) = r^2 - 2r + 2$

$\boxed{y'' - 2y' + 2 = 0}$

Question 8

Find the general solution of the differential equation below using variation of parameters:

$$y'' - 10y' + 25y = \frac{2e^{5t}}{1+t^2}$$

1. solve y_c

$y'' - 10y' + 25y = 0$

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$y_c = c_1 e^{5t} + c_2 t e^{5t}$

$y_1 = e^{5t} \quad y_2 = te^{5t}$

2. V. of P.

$$y_p = y_1 \int \frac{-y_2 F(t)}{W} dt + y_2 \int \frac{y_1 F(t)}{W} dt$$

$$W = \begin{vmatrix} e^{5t} & te^{5t} \\ 5e^{5t} & 5te^{5t} + e^{5t} \end{vmatrix} = e^{5t}(5te^{5t} + e^{5t}) - 5te^{10t}$$

$$= 5te^{10t} + e^{10t} - 5te^{10t} = e^{10t} = W$$

$$y_p = y_1 \int \frac{-y_2 F(t)}{W} dt + y_2 \int \frac{y_1 F(t)}{W} dt$$

$$= e^{5t} \int \frac{-te^{5t}\left(\frac{2e^{5t}}{1+t^2}\right)}{e^{10t}} dt + te^{5t} \int \frac{e^{5t}\left(\frac{2e^{5t}}{1+t^2}\right)}{e^{10t}} dt$$

$$= e^{5t} \int \frac{-2t}{1+t^2} dt + te^{5t} \int \frac{2}{1+t^2} dt$$

(u sub) (tan⁻¹(t))

$$y_p = -e^{5t}\ln(1+t^2) + 2te^{5t}\tan^{-1}(t)$$

3. general solution

$y = y_c + y_p$

$$y = c_1 e^{5t} + c_2 te^{5t} + -e^{5t}\ln(1+t^2) + 2te^{5t}\tan^{-1}(t)$$

Question 9

Determine the general solution $\mathbf{x}(t)$ to the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \mathbf{x}.$$

$$x' = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} x$$

$$\chi(\lambda) = (2-\lambda)(1-\lambda)^2 = 0$$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} -1 & 3 & 3 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1.5 & 0 \\ 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 1.5x_3 = 0 \quad x_1 = -1.5x_3$

$x_2 + 1.5x_3 = 0 \quad x_2 = -1.5x_3$

$$\left\{ \begin{bmatrix} -1.5 \\ -1.5 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

geometric mult. < alg. mult. $\Rightarrow$ NON-DIAGONALIZABLE!

$x_1 = 0$

$x_2 = -x_3$

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \leftarrow \text{Need 1 more} \longrightarrow (A-I)^2 = \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

pick $v_g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$v_1 = (A-I)v_g = \begin{bmatrix} 0 & 3 & 3 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

* $x = Py$ *

$$y = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1.5 & 0 & 1 \\ -1.5 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$x = Py = c_1 e^{2t} \begin{bmatrix} -1.5 & 0 & 1 \\ -1.5 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1.5 & 0 & 1 \\ -1.5 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -1.5 & 0 & 1 \\ -1.5 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$$

$$\boxed{x = c_1 e^{2t} \begin{bmatrix} -1.5 \\ -1.5 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 1 \\ t \\ -t \end{bmatrix}}$$

Question 10

Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}$$

1. Of course, the origin is the only critical point of this system. What kind of critical point is it?
2. Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

$$A = \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}$$

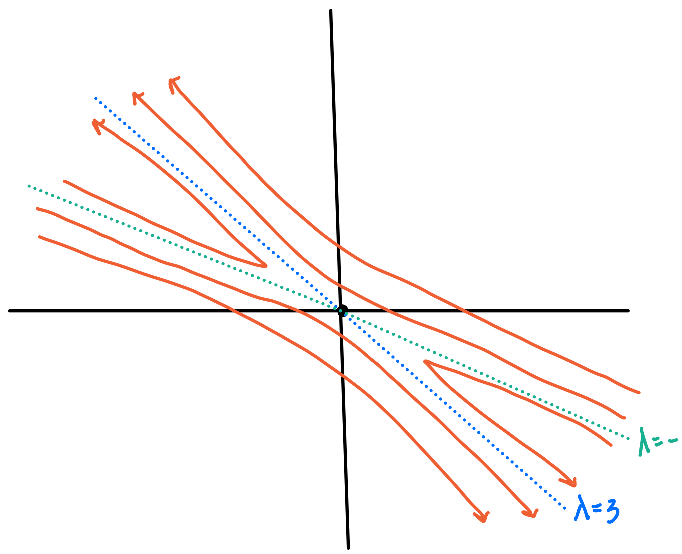
$$\chi(\lambda) = (-5-\lambda)(7-\lambda) + 32$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

$$\lambda = 3, -1$$

critical point is a saddle point



$\lambda = 3$

$$\left[\begin{array}{cc|c} -8 & -8 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$\lambda = -1$

$$\left[\begin{array}{cc|c} -4 & -8 & 0 \\ 4 & 8 & 0 \end{array}\right]$$

$$\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

MATH 2400 – The Big Quiz – Fall 2023

1. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^6$ spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$.

(a) Show that the set of vectors is linearly independent and hence the set is a basis for $\mathcal{V}$. What is dim $\mathcal{V}$?

Call the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, the first component is c_1 and the third component is c_2, and the fifth component is c_3, so if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, we must have $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, so the vectors are linearly independent. Thus dim $\mathcal{V} = 3$.

(b) Is there a 6-by-4 matrix L whose image is $\mathcal{V}$? If there is, give an example. If not, explain why not.

The columns of L should span $\mathcal{V}$ and only $\mathcal{V}$, so: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 3 & -2 & 0 \end{bmatrix}$ works.

(c) Is there a 2-by-6 matrix M whose kernel is $\mathcal{V}$? If there is, give an example. If not, explain why not.

Since, dim $\mathcal{V} = 3$, we would need rk$(M) = 3$ to satisfy the rank/nullity theorem, but rk$(M) \leqslant 2$ since it has only 2 rows.

2. An 8-by-8 matrix A has the following properties:

- The characteristic polynomial of A is $(\lambda - 4)^3(\lambda + 7)^5$
- The nullity of $A - 4\mathbf{I}$ is 1
- The nullity of $A + 7\mathbf{I}$ is 3
- The nullity of $(A + 7\mathbf{I})^2$ is 4

What is the Jordan canonical form of A?

Let J be the Jordan canonical form of A. There are three 4's on the diagonal of J and five -7's.

Since the nullity of $A - 4\mathbf{I}$ is 1, there is only one Jordan block with 4's on its diagonal. Therefore the $\lambda = 4$ part of J consists of a single 3-by-3 Jordan block.

Since the nullity of $A + 7\mathbf{I}$ is 3, there are three linearly independent eigenvectors corresponding to $\lambda = -7$, and so the $\lambda = -7$ part of J has three Jordan blocks. But since the nullity of $(A + 7\mathbf{I})^2$ is 4, only one of these three blocks is bigger than 1-by-1. Therefore

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 \\ 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 0 \end{bmatrix}$.

The characteristic polynomial of A turns out to be $\chi_A = -(\lambda - 9)(\lambda + 3)^3$.

(a) How do you know that A is diagonalizable just by looking at it?

(b) Find a diagonal matrix D and an *orthogonal* matrix P such that $A = PDP^T$.

(a) Because A is symmetric, we know it is diagonalizable.

(b) From χ_A, we see that the eigenvalues of A are 9 (with multiplicity 1) and -3 (with multiplicity 3). Therefore $D = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

For $\lambda = 9$, since all the rows of $A - 9\mathbf{I}$ sum to 0, we conclude that the kernel of $A - 9\mathbf{I}$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = -3$, we can reason that we need three vectors that are orthogonal to $\mathbf{v}_1$ (i.e., that sum to zero), and to each other. So, after normalizing, we get

$$P = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{bmatrix}$$

4. (a) Find the general solution of $x^2y'' - 3xy' + 3y = 0$.

If $y = x^p$ then $x^2y'' - 3xy' + 3y = x^p(p^2 - p - 3p + 3) = x^p(p^2 - 4p + 3)$. The roots of $p^2 - 4p + 3$ are 1 and 3, so the general solution is $y = c_1x + c_2x^3$.

(b) Find the general solution of $x^2y'' - 3xy' + 3y = 4x^2$.

Since x^2 is not a solution of the homogeneous equation, we guess $y_p = Ax^2$. Then $y_p' = 2Ax$ and $y_p'' = 2A$ and so

$$x^2y_p'' - 3xy_p' + 3y_p = -Ax^2$$

which is supposed to be $4x^2$, so $A = -4$ The general solution is

$$y = -4x^2 + c_1x + c_2x^3$$

(c) Solve the initial-value problem $x^2y'' - 3xy' + 3y = 4x^2$, $y(1) = 1$, $y'(1) = 7$.

From the general solution in part (b), we have $y(1) = -4 + c_1 + c_2$ and $y'(0) = -8 + c_1 + 3c_2$. So we need to solve

$$c_1 + c_2 - 4 = 1$$
$$c_1 + 3c_2 - 8 = 7$$

and we get $c_1 = 0$ and $c_2 = 5$. The (unique) solution of the initial-value problem is

$$y = -4x^2 + 5x^3$$

5. Which of the following pairs of functions could be two of the four linearly independent solutions of a homogeneous fourth-order equation with constant coefficients? If there is such an equation, write it down. If there is no such equation, explain why.

(a) $\{t^2 \cos t, t\}$

With $t^2 \cos t$, there must also be $t^2 \sin t$, $t \cos t$, $t \sin t$, $\cos t$ and $\sin t$ and with t there must also be 1. This would require an equation of order at least 8.

(b) $\{te^{2t}, \cos t\}$

With te^{2t} there must also be e^{2t} and with $\cos t$ there must also be $\sin t$. Therefore the auxiliary polynomial is $(r^2+1)(r-2)^2 = (r^2+1)(r^2-4r+4) = r^4 - 4r^3 + 5r^2 - 4r + 4$ and so the equation is $y'''' - 4y''' + 5y'' - 4y' + 4y = 0$.

(c) $\{t^2 e^t, te^{2t}\}$

With $t^2 e^t$ there must also be te^t and e^t, and with te^{2t} there must also be e^{2t}. This would require an equation of order at least 5.

6. (a) Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

The characteristic polynomial of A is $\chi = (\lambda - 2)^2(\lambda^2 - 2\lambda + 5)$, so the eigenvalues of A are 2, 2 and $1 \pm 2i$.

For $\lambda = 2$ there is a single eigenvector, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the generalized eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda = 1 + 2i$ we have the eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$ and

$$e^t(\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} = e^t \left(\begin{bmatrix} 0 \\ 0 \\ \cos 2t \\ -\sin 2t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \sin 2t \\ \cos 2t \end{bmatrix} \right)$$

so the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ 0 \\ \cos 2t \\ -\sin 2t \end{bmatrix} + c_4 e^t \begin{bmatrix} 0 \\ 0 \\ \sin 2t \\ \cos 2t \end{bmatrix}$$

(b) Solve the initial-value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 6 \end{bmatrix}$ for the matrix A in part (a).

Since for our solution from part (a), $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$, the solution of the initial-value problem is

$$\mathbf{x} = 3e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} t \\ 1 \\ 0 \\ 0 \end{bmatrix} + 5e^{t} \begin{bmatrix} 0 \\ 0 \\ \cos 2t \\ -\sin 2t \end{bmatrix} + 6e^{t} \begin{bmatrix} 0 \\ 0 \\ \sin 2t \\ \cos 2t \end{bmatrix}$$

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}$

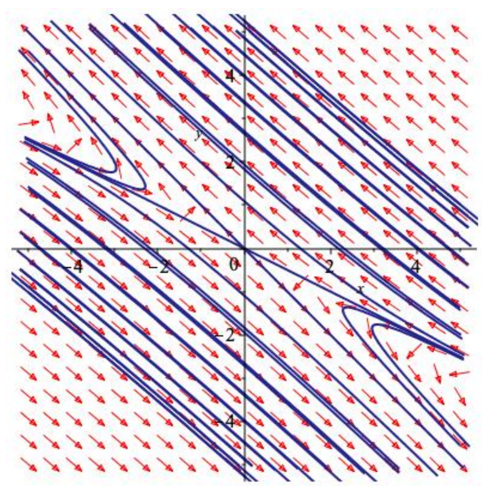
(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since the det$A = -3 < 0$, the origin is a(n unstable) saddle point.

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

We need the eigenvalues and eigenvectors of A. The characteristic polynomial of A is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$, so its eigenvalues are -1 and 3.

For $\lambda = -1$, an eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and for $\lambda = 3$, an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. There are half-line trajectories along the eigenvectors; those in the direction of $\pm \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ will approach the origin and those in the direction of $\pm \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will move away. The other curves will come in toward the origin along $\pm \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and then turn and go out along $\pm \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, becoming tangent to the straight trajectories as $t \to \pm\infty$:



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x^3 - x - y$$

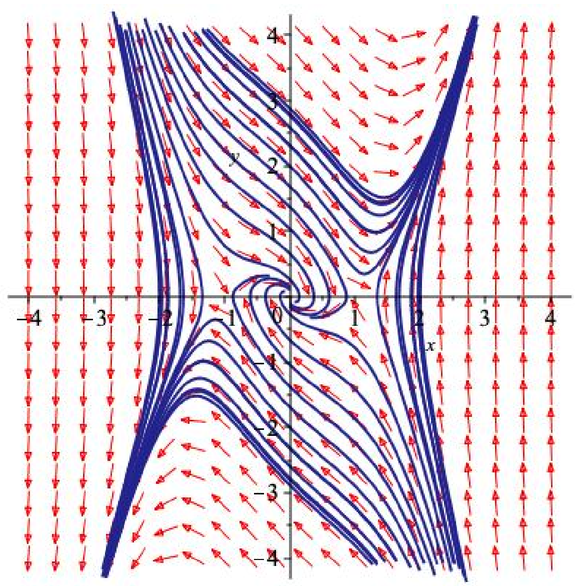
For the critical points, we need both right-hand sides to be zero. For the first one we need $y = 0$. And then the second is zero if $x = 0, 1, -1$. Therefore there are three critical points $(-1, 0)$, $(0, 0)$ and $(1, 0)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & -1 \end{bmatrix}$.

At $(0, 0)$ we have $J(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, which is a stable spiral.

At $(\pm 1, 0)$, $J(\pm 1, 0) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, so both of these are saddle points.

For the record:



MATH 2400 – The Big Quiz – Fall 2023

1. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^6$ spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Show that the set of vectors is linearly independent and hence the set is a basis for $\mathcal{V}$. What is dim $\mathcal{V}$?

Call the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. In the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the first component is $c_1 + 6c_2$ and the third component is $6c_2 + c_2$, so if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, we must have $c_1 + 6c_2 = 0$ and $6c_1 + c_2 = 0$, which implies $c_1 = c_2 = 0$, so the vectors are linearly independent. Thus dim $\mathcal{V} = 2$.

(b) Is there a 6-by-3 matrix L whose image is $\mathcal{V}$? If there is, give an example. If not, explain why not.

The columns of L should span $\mathcal{V}$ and only $\mathcal{V}$, so: $\begin{bmatrix} 1 & 6 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & 0 \\ 4 & 3 & 0 \\ 5 & 2 & 0 \\ 6 & 1 & 0 \end{bmatrix}$ works.

(c) Is there a 3-by-6 matrix M whose kernel is $\mathcal{V}$? If there is, give an example. If not, explain why not.

Since dim ker(A) = dim $\mathcal{V} = 2$, we would need rk$(M) = 4$ to satisfy the rank/nullity theorem, but rk$(M) \leqslant 3$ since it has only 3 rows.

2. An 8-by-8 matrix A has the following properties:

- The characteristic polynomial of A is $(\lambda - 2)^4(\lambda + 3)^4$
- The nullity of $A - 2\mathbf{I}$ is 3
- The nullity of $A + 3\mathbf{I}$ is 2
- The nullity of $(A + 3\mathbf{I})^2$ is 4

What is the Jordan canonical form of A?

Let J be the Jordan canonical form of A. There are four 2's on the diagonal of J and four -3's.

Since the nullity of $A - 2\mathbf{I}$ is 3, there are three Jordan block with 2's on its diagonal. Therefore the $\lambda = 2$ part of J consists of one 2-by-2 Jordan block and two 1-by-1 blocks.

Since the nullity of $A + 3\mathbf{I}$ is 2, there are two linearly independent eigenvectors corresponding to $\lambda = -3$, and so the $\lambda = -3$ part of J has two Jordan blocks. But since the nullity of $(A + 3\mathbf{I})^2$ is 4, both of these three blocks are bigger than 1-by-1. Therefore

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix}$.

The characteristic polynomial of A turns out to be $\chi_A = -(\lambda - 6)(\lambda + 2)^3$.

(a) How do you know that A is diagonalizable just by looking at it?

(b) Find a diagonal matrix D and an *orthogonal* matrix P such that $A = PDP^T$.

(a) Because A is symmetric, we know it is diagonalizable.

(b) From χ_A, we see that the eigenvalues of A are 6 (with multiplicity 1) and -2 (with multiplicity 3). Therefore $D = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

For $\lambda = 9$, since all the rows of $A - 6\mathbf{I}$ sum to 0, we conclude that the kernel of $A - 6\mathbf{I}$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = -2$, we can reason that we need three vectors that are orthogonal to $\mathbf{v}_1$ (i.e., that sum to zero), and to each other. So, after normalizing, we get

$$P = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{bmatrix}$$

4. (a) Find the general solution of $x^2y'' - 4xy' + 4y = 0$.

If $y = x^p$ then $x^2y'' - 4xy' + 4y = x^p(p^2 - p - 4p + 4) = x^p(p^2 - 5p + 4)$. The roots of $p^2 - 5p + 4$ are 1 and 4, so the general solution is $y = c_1x + c_2x^4$.

(b) Find the general solution of $x^2y'' - 4xy' + 4y = 4x^2$.

Since x^2 is not a solution of the homogeneous equation, we guess $y_p = Ax^2$. Then $y_p' = 2Ax$ and $y_p'' = 2A$ and so

$$x^2y_p'' - 4xy_p' + 4y_p = -Ax^2$$

which is supposed to be $4x^2$, so $A = -2$ The general solution is

$$y = -2x^2 + c_1x + c_2x^4$$

(c) Solve the initial-value problem $x^2y'' - 4xy' + 4y = 4x^2$, $y(1) = 4$, $y'(1) = 8$.

From the general solution in part (b), we have $y(1) = -2 + c_1 + c_2$ and $y'(0) = -4 + c_1 + 4c_2$. So we need to solve

$$c_1 + c_2 - 2 = 4$$
$$c_1 + 4c_2 - 4 = 8$$

and we get $c_1 = 4$ and $c_2 = 2$. The (unique) solution of the initial-value problem is

$$y = -2x^2 + 4x + 2x^4$$

5. Which of the following pairs of functions could be two of the four linearly independent solutions of a homogeneous fourth-order equation with constant coefficients? If there is such an equation, write it down. If there is no such equation, explain why.

(a) $\{t^2 \sin t, e^t\}$

With $t^2 \sin t$, there must also be $t^2 \cos t$, $t \cos t$, $t \sin t$, $\cos t$ and $\sin t$ and with e^t, this would require an equation of order at least 7.

(b) $\{t^2 e^{2t}, \cos t\}$

With $t^2 e^{2t}$ there must also be te^{2t} and e^{2t}, and with $\cos t$ there must also be $\sin t$. This would require an equation of order at least 5.

(c) $\{te^t, te^{2t}\}$

With te^t there must also be e^t and with te^{2t} there must also be e^{2t}. Therefore the auxiliary polynomial is $(r-1)^2(r-2)^2 = (r^2 - 2r + 1)(r^2 - 4r + 4) = r^4 - 6r^3 + 13r^2 - 12r + 4$ and so the equation is $y'''' - 6y''' + 13y'' - 12y' + 4y = 0$.

6. (a) Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

The characteristic polynomial of A is $\chi = (\lambda - 3)^2(\lambda^2 - 4\lambda + 5)$, so the eigenvalues of A are 3, 3 and $2 \pm i$.

For $\lambda = 3$ there is a single eigenvector, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the generalized eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda = 2 + i$ we have the eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$ and

$$e^{2t}(\cos t + i \sin t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} = e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ \cos t \\ -\sin t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \sin t \\ \cos t \end{bmatrix} \right)$$

so the general solution is

$$\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} t \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 0 \\ \cos t \\ -\sin t \end{bmatrix} + c_4 e^{2t} \begin{bmatrix} 0 \\ 0 \\ \sin t \\ \cos t \end{bmatrix}$$

(b) Solve the initial-value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 7 \\ 1 \end{bmatrix}$ for the matrix A in part (a).

Since for our solution from part (a), $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$, the solution of the initial-value problem is

$$\mathbf{x} = 2e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 7e^{t}\begin{bmatrix} 0 \\ 0 \\ \cos 2t \\ -\sin 2t \end{bmatrix} + e^{t}\begin{bmatrix} 0 \\ 0 \\ \sin 2t \\ \cos 2t \end{bmatrix}$$

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

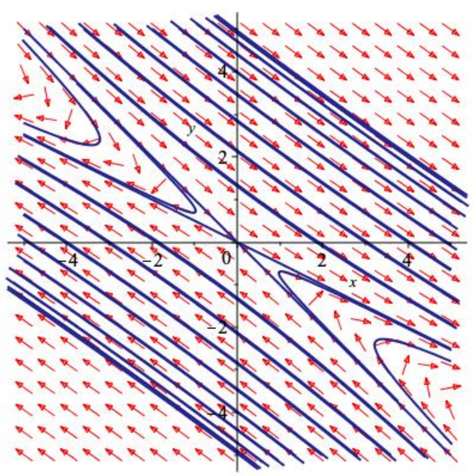
(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since the det$A = -1 < 0$, the origin is a(n unstable) saddle point.

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

We need the eigenvalues and eigenvectors of A. The characteristic polynomial of A is $\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$, so its eigenvalues are 1 and -1.

For $\lambda = 1$, an eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and for $\lambda = -1$, an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. There are half-line trajectories along the eigenvectors; those in the direction of $\pm\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ will move away from the origin and those in the direction of $\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will approach the origin. The other curves will come in toward the origin along $\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and then turn and go out along $\pm\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, becoming tangent to the straight trajectories as $t \to \pm\infty$:



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x^3 - x + y$$

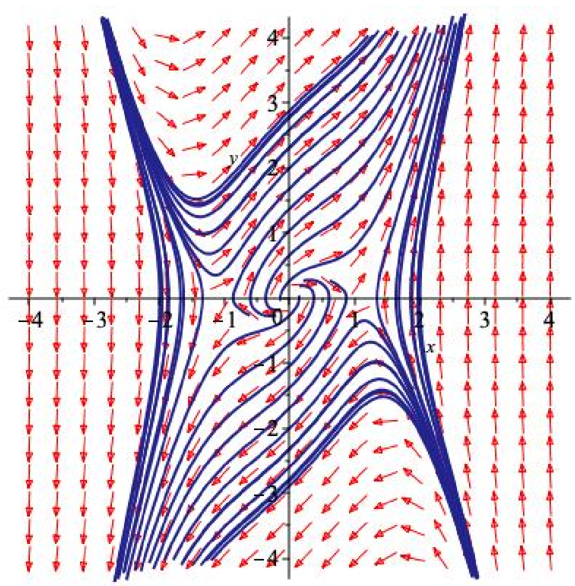
For the critical points, we need both right-hand sides to be zero. For the first one we need $y = 0$. And then the second is zero if $x = 0, 1, -1$. Therefore there are three critical points $(-1, 0)$, $(0, 0)$ and $(1, 0)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 1 \end{bmatrix}$.

At $(0, 0)$ we have $J(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, which is an unstable spiral.

At $(\pm 1, 0)$, $J(\pm 1, 0) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, so both of these are saddle points.

For the record:



MATH 2400 – The Big Quiz – Fall 2023

1. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^6$ spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Show that the set of vectors is linearly independent and hence the set is a basis for $\mathcal{V}$. What is $\dim \mathcal{V}$?

Call the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, the first component is $c_1 + c_3$, the second component is c_2 and the third component is c_3, so if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, we must have $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, so the vectors are linearly independent. Thus $\dim \mathcal{V} = 3$.

(b) Is there a 6-by-2 matrix L whose image is $\mathcal{V}$? If there is, give an example. If not, explain why not.

Since the matrix has only two columns, its rank is at most 2, which is less than $\dim \mathcal{V}$, so there is no such matrix.

(c) Is there a 4-by-6 matrix M whose kernel is $\mathcal{V}$? If there is, give an example. If not, explain why not.

Yes. We need the rank of M to be 3, so we need three linearly independent vectors perpendicular to the three vectors in the basis of $\mathcal{V}$ to be three non-zero rows of M. For instance

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. An 8-by-8 matrix A has the following properties:

- The characteristic polynomial of A is $(\lambda - 5)^6(\lambda + 2)^2$
- The nullity of $A - 5\mathbf{I}$ is 3
- The nullity of $(A - 5\mathbf{I})^2$ is 5
- The nullity of $A + 2\mathbf{I}$ is 1

What is the Jordan canonical form of A?

Let J be the Jordan canonical form of A. There are six 5's on the diagonal of J and two -2's.

Since the nullity of $A - 5\mathbf{I}$ is 3, there are three Jordan blocks with 5's on its diagonal. Therefore the $\lambda = 5$ part of J consists of three Jordan blocks. Since the nullity of $(A - 5\mathbf{I})^2$ is 5, two of the blocks is bigger than 1-by-1. So there's a 3-by-3 block, a 2-by-2 block, and a 1-by-1 block.

Since the nullity of $A + 2\mathbf{I}$ is 1, there is only one linearly independent eigenvector corresponding to $\lambda = -2$, and so the $\lambda = -2$ part of J has one Jordan block. Therefore

$$J = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 0 & -3 & -3 & -3 \\ -3 & 0 & -3 & -3 \\ -3 & -3 & 0 & -3 \\ -3 & -3 & -3 & 0 \end{bmatrix}$.

The characteristic polynomial of A turns out to be $\chi_A = -(\lambda+9)(\lambda-3)^3$.

(a) How do you know that A is diagonalizable just by looking at it?

(b) Find a diagonal matrix D and an *orthogonal* matrix P such that $A = PDP^T$.

(a) Because A is symmetric, we know it is diagonalizable.

(b) From χ_A, we see that the eigenvalues of A are -9 (with multiplicity 1) and 3 (with multiplicity 3). Therefore $D = \begin{bmatrix} -9 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

For $\lambda = -9$, since all the rows of $A + 9\mathbf{I}$ sum to 0, we conclude that the kernel of $A + 9\mathbf{I}$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 3$, we can reason that we need three vectors that are orthogonal to $\mathbf{v}_1$ (i.e., that sum to zero), and to each other. So, after normalizing, we get

$$P = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{bmatrix}$$

4. (a) Find the general solution of $x^2y'' - 8xy' + 8y = 0$.

If $y = x^p$ then $x^2y'' - 8xy' + 8y = x^p(p^2 - p - 8p + 8) = x^p(p^2 - 9p + 8)$. The roots of $p^2 - 9p + 8$ are 1 and 8, so the general solution is $y = c_1x + c_2x^8$.

(b) Find the general solution of $x^2y'' - 8xy' + 8y = 6x^2$.

Since x^2 is not a solution of the homogeneous equation, we guess $y_p = Ax^2$. Then $y_p' = 2Ax$ and $y_p'' = 2A$ and so

$$x^2y_p'' - 8xy_p' + 8y_p = -6Ax^2$$

which is supposed to be $6x^2$, so $A = -1$ The general solution is

$$y = -x^2 + c_1x + c_2x^8$$

(c) Solve the initial-value problem $x^2y'' - 8xy' + 8y = 6x^2$, $y(1) = 1$, $y'(1) = 7$.

From the general solution in part (b), we have $y(1) = -1 + c_1 + c_2$ and $y'(0) = -2 + c_1 + 8c_2$. So we need to solve

$$c_1 + c_2 - 1 = 1$$
$$c_1 + 8c_2 - 2 = 7$$

and we get $c_1 = 1$ and $c_2 = 1$. The (unique) solution of the initial-value problem is

$$y = -x^2 + x + x^3$$

5. Which of the following pairs of functions could be two of the four linearly independent solutions of a homogeneous fourth-order equation with constant coefficients? If there is such an equation, write it down. If there is no such equation, explain why.

(a) $\{t \sin t, te^t\}$

With $t \sin t$, there must also be , $t \cos t$, $\cos t$ and $\sin t$ and with te^t, there must also be e^t — this would require an equation of order at least 6.

(b) $\{t^2e^{2t}, t^2e^t\}$

With t^2e^{2t} there must also be te^{2t} and e^{2t}, sand with t^2e^t there must also be te^t and e^t. This would require an equation of order at least 6.

(c) $\{te^t, te^{2t}\}$

With te^t there must also be e^t and with te^{2t} there must also be e^{2t}. Therefore the auxiliary polynomial is $(r-1)^2(r-2)^2 = (r^2 - 2r + 1)(r^2 - 4r + 4) = r^4 - 6r^3 + 13r^2 - 12r + 4$ and so the equation is $y'''' - 6y''' + 13y'' - 12y' + 4y = 0$.

6. (a) Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 4 & 3 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The characteristic polynomial of A is $\chi = (\lambda - 1)^2(\lambda^2 - 8\lambda + 25)$, so the eigenvalues of A are 1, 1 and $4 \pm 3i$.

For $\lambda = 4$ there is a single eigenvector, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the generalized eigenvector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 4 + 3i$ we have the eigenvector $\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}$ and

$$e^{4t}(\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} = e^{4t} \left(\begin{bmatrix} \cos 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} \sin 3t \\ \cos 3t \\ 0 \\ 0 \end{bmatrix} \right)$$

so the general solution is

$$\mathbf{x} = c_1 e^{4t} \begin{bmatrix} \cos 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin 3t \\ \cos 3t \\ 0 \\ 0 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 e^{t} \begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix}$$

(b) Solve the initial-value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}$ for the matrix A in part (a).

Since for our solution from part (a), $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$, the solution of the initial-value problem is

$$\mathbf{x} = 2e^{4t}\begin{bmatrix} \cos 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix} + 3e^{4t}\begin{bmatrix} \sin 3t \\ \cos 3t \\ 0 \\ 0 \end{bmatrix} + 4e^{t}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 6e^{t}\begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix}$$

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 4 & 6 \\ -3 & -5 \end{bmatrix}$

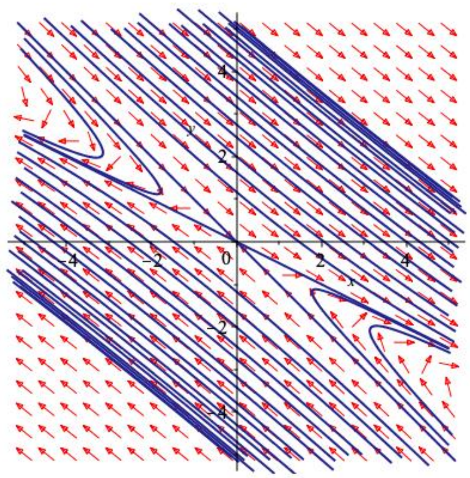
(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since the det$A = -2 < 0$ is lower triangular, the origin is a(n unstable) saddle point.

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

We need the eigenvalues and eigenvectors of A. The characteristic polynomial of A is $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$, so its eigenvalues are 1 and -2.

For $\lambda = 1$, an eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and for $\lambda = -2$, an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. There are half-line trajectories along the eigenvectors; those in the direction of $\pm\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ will move away from the origin and those in the direction of $\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will approach the origin. The other curves will come in toward the origin along $\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and then turn and go out along $\pm\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, becoming tangent to the straight trajectories as $t \to \pm\infty$:



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = y^3 - y + x$$

$$\frac{dy}{dt} = x$$

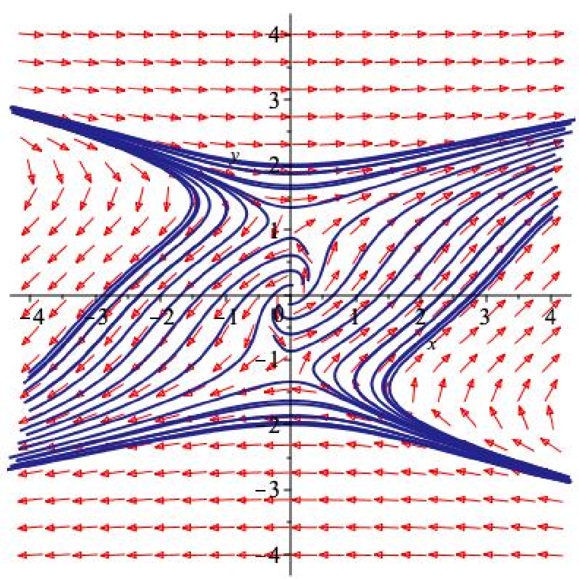
For the critical points, we need both right-hand sides to be zero. For the second one we need $x = 0$. And then the first is zero if $y = 0, 1, -1$. Therefore there are three critical points $(0, -1)$, $(0, 0)$ and $(0, 1)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} 1 & 3y^2 - 1 \\ 1 & 0 \end{bmatrix}$.

At $(0, 0)$ we have $J(0, 0) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, which is an unstable spiral.

At $(0, \pm 1)$, $J(0, \pm 1) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, so both of these are saddle points.

For the record:



MATH 2400 – The Big Quiz – Fall 2023

1. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^6$ spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Show that the set of vectors is linearly independent and hence the set is a basis for $\mathcal{V}$. What is $\dim \mathcal{V}$?

Call the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, the first component is c_1, the second component is c_2 and the third component is c_3, so if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, we must have $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, so the vectors are linearly independent. Thus $\dim \mathcal{V} = 3$.

(b) Is there a 6-by-5 matrix L whose image is $\mathcal{V}$? If there is, give an example. If not, explain why not.

The columns of L should span $\mathcal{V}$ and only $\mathcal{V}$, so: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ works.

(c) Is there a 5-by-6 matrix M whose kernel is $\mathcal{V}$? If there is, give an example. If not, explain why not.

Yes. We need the rank of M to be 3, so we need three linearly independent vectors perpendicular to the three vectors in the basis of $\mathcal{V}$ to be three non-zero rows of M. For instance

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. An 8-by-8 matrix A has the following properties:

- The characteristic polynomial of A is $(\lambda - 1)^2(\lambda + 9)^6$
- The nullity of $A - \mathbf{I}$ is 2
- The nullity of $A + 9\mathbf{I}$ is 2
- The nullity of $(A + 9\mathbf{I})^2$ is 3

What is the Jordan canonical form of A?

Let J be the Jordan canonical form of A. There are six 1's on the diagonal of J and six -9's.

Since the nullity of $A - 2\mathbf{I}$ is 2, there are two Jordan blocks with 1's on their diagonal. Therefore the $\lambda = 1$ part of J consists of two 1-by-1 Jordan blocks.

Since the nullity of $A + 9\mathbf{I}$ is 2, there are two linearly independent eigenvectors corresponding to $\lambda = -9$, and so the $\lambda = -9$ part of J has two Jordan blocks. But since the nullity of $(A + 9\mathbf{I})^2$ is 3, only one of these three blocks is bigger than 1-by-1. Therefore

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 0 & -2 & -2 & -2 \\ -2 & 0 & -2 & -2 \\ -2 & -2 & 0 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix}$.

The characteristic polynomial of A turns out to be $\chi_A = -(\lambda + 6)(\lambda - 2)^3$.

(a) How do you know that A is diagonalizable just by looking at it?

(b) Find a diagonal matrix D and an *orthogonal* matrix P such that $A = PDP^T$.

(a) Because A is symmetric, we know it is diagonalizable.

(b) From χ_A, we see that the eigenvalues of A are -6 (with multiplicity 1) and 2 (with multiplicity 3). Therefore $D = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

For $\lambda == 6$, since all the rows of $A + 6\mathbf{I}$ sum to 0, we conclude that the kernel of $A + 6\mathbf{I}$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, we can reason that we need three vectors that are orthogonal to $\mathbf{v}_1$ (i.e., that sum to zero), and to each other. So, after normalizing, we get

$$P = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{bmatrix}$$

4. (a) Find the general solution of $x^2y'' - 6xy' + 6y = 0$.

If $y = x^p$ then $x^2y'' - 3xy' + 3y = x^p(p^2 - p - 6p + 6) = x^p(p^2 - 7p + 6)$. The roots of $p^2 - 7p + 6$ are 1 and 6, so the general solution is $y = c_1x + c_2x^6$.

(b) Find the general solution of $x^2y'' - 6xy' + 6y = 4x^2$.

Since x^2 is not a solution of the homogeneous equation, we guess $y_p = Ax^2$. Then $y_p' = 2Ax$ and $y_p'' = 2A$ and so

$$x^2y_p'' - 6xy_p' + 6y_p = -Ax^2$$

which is supposed to be $4x^2$, so $A = -1$ The general solution is

$$y = -x^2 + c_1x + c_2x^6$$

(c) Solve the initial-value problem $x^2y'' - 6xy' + 6y = 4x^2$, $y(1) = 4$, $y'(1) = 8$.

From the general solution in part (b), we have $y(1) = -1 + c_1 + c_2$ and $y'(0) = -2 + c_1 + 6c_2$. So we need to solve

$$c_1 + c_2 - 1 = 4$$
$$c_1 + 6c_2 - 2 = 8$$

and we get $c_1 = 4$ and $c_2 = 1$. The (unique) solution of the initial-value problem is

$$y = -x^2 + 4x + x^6$$

5. Which of the following pairs of functions could be two of the four linearly independent solutions of a homogeneous fourth-order equation with constant coefficients? If there is such an equation, write it down. If there is no such equation, explain why.

(a) $\{t^2, e^t\}$

With t^2, there must also be t and 1, so the auxiliary polynomial is $r^3(r-1) = r^4 - r^3$. Therefore the equation is $y'''' - y''' = 0$.

(b) $\{te^{2t}, \cos t\}$

With te^{2t} there must also be e^{2t}, and with $\cos t$ there must also be $\sin t$. Therefore the auxiliary polynomial is $(r-2)^2(r^2+1) = (r^2-4r+4)(r^2+1) = r^4 - 4r^3 + 5r^2 - 4r + 4$. Therefore the equation is $y'''' - 4y''' + 5y'' - 4y' + 4y = 0$.

(c) $\{t^3e^t, e^{2t}\}$

With t^3e^t there must also be t^2e^t, te^t and e^t. So we now have five independent solutions and the equation would have to be at least fifth order.

6. (a) Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

The characteristic polynomial of A is $\chi = (\lambda - 2)^2(\lambda^2 - 2\lambda + 10)$, so the eigenvalues of A are 2, 2 and $1 \pm 3i$.

For $\lambda = 2$ there is a single eigenvector, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the generalized eigenvector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 1 + 3i$ we have the eigenvector $\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}$ and

$$e^t(\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} = e^t \left(\begin{bmatrix} \cos 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} \sin 3t \\ \cos 3t \\ 0 \\ 0 \end{bmatrix} \right)$$

so the general solution is

$$\mathbf{x} = c_1 e^t \begin{bmatrix} \cos 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 3t \\ \cos 3t \\ 0 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 e^{2t} \begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix}$$

(b) Solve the initial-value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 8 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ for the matrix A in part (a).

Since for our solution from part (a), $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$, the solution of the initial-value problem is

$$\mathbf{x} = 8e^t \begin{bmatrix} \cos 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix} + 5e^t \begin{bmatrix} \sin 3t \\ \cos 3t \\ 0 \\ 0 \end{bmatrix} + 3e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix}$$

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -6 & -8 \\ 4 & 6 \end{bmatrix}$

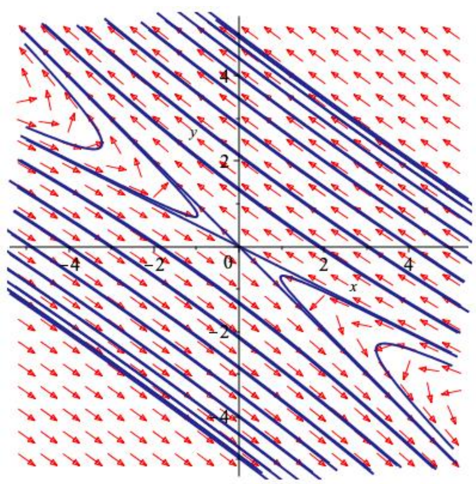
(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since the $\det A = -4 < 0$, the origin is a(n unstable) saddle point.

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

We need the eigenvalues and eigenvectors of A. The characteristic polynomial of A is $\lambda^2 - 4 = (\lambda - 3)(\lambda + 1)$, so its eigenvalues are 2 and -2.

For $\lambda = -2$, an eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and for $\lambda = 2$, an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. There are half-line trajectories along the eigenvectors; those in the direction of $\pm\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ will approach the origin and those in the direction of $\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will move away. The other curves will come in toward the origin along $\pm\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and then turn and go out along $\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, becoming tangent to the straight trajectories as $t \to \pm\infty$:



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = y^3 - y - x$$

$$\frac{dy}{dt} = x$$

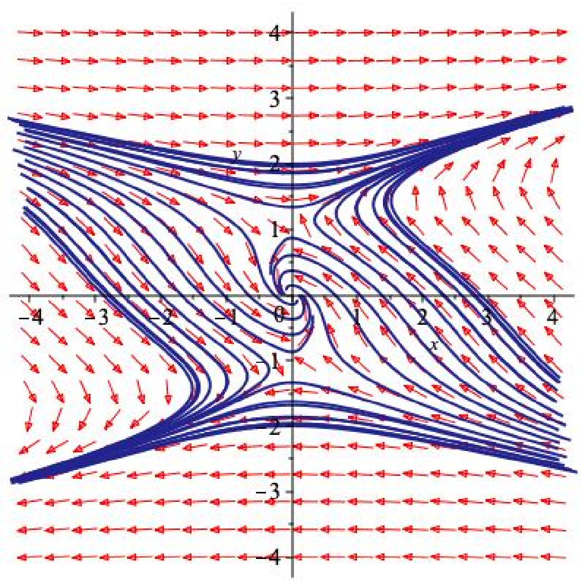
For the critical points, we need both right-hand sides to be zero. For the second one we need $x = 0$. And then the first is zero if $y = 0, 1, -1$. Therefore there are three critical points $(0, -1)$, $(0, 0)$ and $(0, 1)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} -1 & 3y^2 - 1 \\ 1 & 0 \end{bmatrix}$.

At $(0, 0)$ we have $J(0, 0) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, which is a stable spiral.

At $(0, \pm 1)$, $J(0, \pm 1) = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$, so both of these are saddle points.

For the record:



Name: **Penn ID:**

Math 2400 Final Exam 05/04/2023

1. Let A be the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

   (a) Find a basis for the row space of A, i.e. for the span of the rows of A.

   (b) Find a basis for the image of A, i.e. the column space of A.

   (c) Find a basis for $\ker(A)$.

   (d) Is the vector $\begin{bmatrix} 0 & 3 & 1 & 2 & 0 \end{bmatrix}$ in the row space of A? Prove or disprove.

Solutions

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_4 - R_5]{R_1 \to R_1 - R_4 - R_5} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{the rows}]{\text{reordering}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_3]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) From the row reduced echelon form, we see that a basis for $\text{row}(A)$ is given by:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix} \right\}.$$

(b) From the row reduced echelon form, we see that a basis for $\text{col}(A)$ is given by:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) From the row reduced echelon form we get that the solution of $A\vec{x} = 0$ is

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_4 \text{ and } x_5 \text{ free variables} \end{cases}$$

and thus the basis of $\ker(A)$ is given by: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) Yes as $\begin{bmatrix} 0 & 3 & 1 & 2 & 0 \end{bmatrix} = 3\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix} \in \text{row}(A)$.

2. For each following questions, answer True or False. **You need to justify your answer**.

   (a) A linear system with fewer unknowns than equations must have either infinitely many solutions or no solutions.

   (b) If a vector u is a linear combination of vectors v and w, and v is a linear combination of vectors p, q and r, then u must be a linear combination of p, q, r and w.

   (c) For any choice of vectors v_1, v_2 and v_3 in $\mathbb{R}^5$, the subset

$$\{v_1 + 2v_2,\ v_1 - v_2 + 6v_3,\ 4v_2 + v_3,\ v_2 + v_3\}$$

   of $\mathbb{R}^5$ can never be linearly independent.

   (d) There exists an invertible 10×10 matrix that has 92 entries that are precisely equal to 1.

   (e) There exists a 3×3 matrix P such that the linear transformation $T\colon \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ defined by $T(A) = AP - PA$ has rank 9.

Solutions

(a) $\boxed{\text{False}}$ The following system has a unique solution $\begin{cases} x = 2 \\ y = 3 \\ x + y = 5 \end{cases}$

(b) $\boxed{\text{True}}$ If $u = \lambda_1 v + \lambda_2 w$ where $\lambda_1, \lambda_2 \in \mathbb{R}$, and $v = \mu_1 p + \mu_2 q + \mu_3 r$ with $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$, then we get

$$u = \lambda_1\mu_1 p + \lambda_1\mu_2 q + \lambda_1\mu_3 r + \lambda_2 w$$

(c) $\boxed{\text{True}}$ Any 4 vectors in the subspace $\mathrm{Span}(v_1, v_2, v_3) \subseteq \mathbb{R}^5$ must be linearly dependent.

(d) $\boxed{\text{False}}$ There must be at least 2 rows with only 1 entries, and thus by row reduction, we obtain a line of only zero entries, hence the matrix has determinant zero and thus cannot be invertible.

(e) $\boxed{\text{False}}$ We have $T(I_3) = P - P = 0$, thus $I_3 \in \ker(T)$, hence $\mathrm{nullity}(T) \geq 1$. By rank-nullity, we get $\mathrm{rank}(T) \leq 8$.

3. (a) Find the general solution of the homogeneous linear system: $\begin{cases} x'(t) = 4x(t) - 2y(t) \\ y'(t) = -2x(t) + 4y(t) \end{cases}$

   (b) Determine which figure below represents the trajectories of the solutions of the system. You must justify your answer.

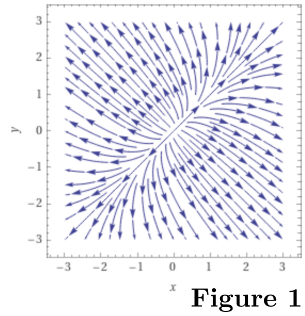


Figure 1

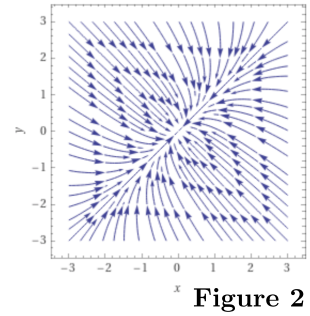


Figure 2

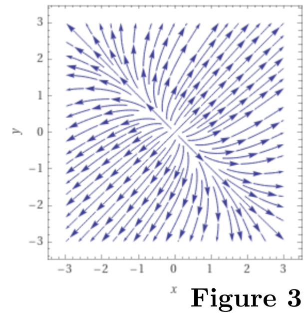


Figure 3

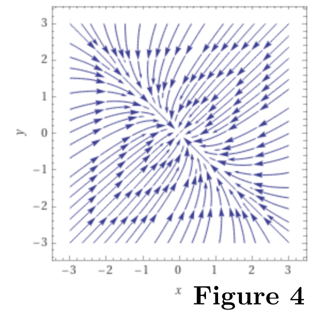


Figure 4

Solutions

(a) This system is of the form $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, where $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$. The rows add up to 2 thus $\lambda_1 = 2$ is an eigenvalue of A with eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since the trace is sum of eigenvalues, the other eigenvalue is $\lambda_2 = 6$. Since A is symmetric, the eigenvector v_2 of λ_2 is orthogonal to v_1. We can pick $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus general solution of the system is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) Since the eigenvalues are distinct and positive, the trajectories form a source node, and as v_2 has the largest eigenvalue, the trajectories are parallel to v_2 overtime, thus is must be $\boxed{\text{Figure 1}}$.

4. Find the general solution to the ordinary differential equation:

$$f'''(t) - 2f''(t) + f'(t) = 4t + 2e^{2t}.$$

Solution First find the homogeneous solution. The characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0.$$

We factor and obtain the roots.

$$\lambda(\lambda^2 - 2\lambda + 1) = \lambda(\lambda - 1)^2 = 0 \implies \lambda = 0,\ 1 \text{ (mult 2)}$$

This generates the following general solution of for the homogeneous equation

$$C_1 e^{0t} + C_2 e^{1t} + C_3 t e^{1t} = C_1 + C_2 e^t + C_3 t e^t$$

We guess a solution to the non-homogeneous equation. The second term as an exponential dictates a guess of the form Ae^{2t}, and the first term is a degree one polynomial which requires a guess of the form $Bt + C$. The constant term is in the homogeneous solution so instead we must guess $t(Bt + C) = Bt^2 + Ct$. Due to the linearity we may just add the guesses for each term. Therefore our particular solution will have the form

$$f_p(t) = Ae^{2t} + Bt^2 + Ct,$$

From which we see that

$$f_p'(t) = 2Ae^{2t} + 2Bt + C, \quad f_p''(t) = 4Ae^{2t} + 2B, \quad f_p'''(t) = 8Ae^{2t}.$$

Substituting into the ODE gives

$$8Ae^{2t} - 8Ae^{2t} - 4B + 2Ae^{2t} + 2Bt + C = 4t + 2e^{2t}.$$

Equating coefficients of like functions gives the system

$$\begin{cases} 2A = 2 \\ 2B = 4 \\ C - 4B = 0 \end{cases}$$

which has the solution $A = 1, B = 2, C = 8$. The any solution to the non-homogeneous ODE can be written as a particular solution plus some homogeneous solution. Therefore the general solution is

$$\boxed{C_1 + C_2 e^t + C_3 t e^t + e^{2t} + 2t^2 + 8t}$$

5. Let $\mathcal{M}_{2\times 2}$ be the vector space of 2×2-matrices. Let $T\colon \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be the linear transformation defined as follows:

$$T(A) = \begin{bmatrix} \mathsf{tr}(A) & 0 \\ 0 & \mathsf{tr}(A) \end{bmatrix}$$

where $\mathsf{tr}(A)$ is the trace of a matrix A.

(a) Compute $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$.

(b) Find the matrix of T with respect to the standard basis of $\mathcal{M}_{2\times 2}$.

(c) Find a basis for the kernel of T (be sure to express the basis in terms of elements in $\mathcal{M}_{2\times 2}$ and not as column vectors).

(d) What are the eigenvalues of T?

(e) Is T diagonalizable? If yes, find a basis in $\mathcal{M}_{2\times 2}$ that diagonalizes T (be sure to express the basis in terms of elements in $\mathcal{M}_{2\times 2}$ and not as column vectors). If not, can you find a Jordan canonical form?

Solutions

(a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) We evaluate T on the standard basis

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we obtain:

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(c) We can either row reduce $[T]$, or alternatively, from (b), we already know that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are in $\ker(T)$ and it's easy to guess that $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is also in the $\ker(T)$. As the image of T is non-trivial, we know the nullity of T is at most 3, and thus actually equals 3, and we obtain that the basis of $\ker(T)$ is $\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

(d) From question (a), we know $\lambda = 2$ is an eigenvalue with eigenvector $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. From question (c), we know $\lambda = 0$ is an eigenvalue with multiplicity of at least 3 (with eigenvectors the basis of $\ker(T)$). Since $[T] \in \mathcal{M}_{4\times 4}$, there can not be more eigenvalues.

(e) Yes T is diagonalizable as the algebraic multiplicity of each eigenvalue equals the geometric multiplicity (the dimension of $\ker([T] - \lambda I_4)$). The diagonalizing basis are the vectors (a) and (c) which are:

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

6. (a) Let $\mathcal{P}_4$ be the vector space of polynomials of degree less than or equal to 4. Among the following subsets of $\mathcal{P}_4$, only one is **not** a subspace, which one is it? You must justify your answer. *You do not need to explain why the others are subspaces.*

(i) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(7) = 0\}$ (ii) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(0) = 0\}$
(iii) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(x) = p(-x)\}$ (iv) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p'(x) \neq 0\}$

(b) Let $\mathcal{M}_2$ be the vector space 2×2-matrices. Among the following functions $T\colon \mathcal{M}_2 \to \mathbb{R}$, only one does **not** define a linear transformation, which one is it? You must justify your answer. *You do not need to explain why the others are linear.*

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + 3b - c + 5d$ (ii) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)^2 + 1 - (a+b+1)^2$
(iii) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$ (iv) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

(c) Among the following systems, only one is **not** autonomous, which one is it? You must justify your answer. *You do not need to explain why the others are autonomous.*

(i) $\begin{cases} x'(t) = x(t) - 2e^t y(t) \\ y'(t) = x^2(t) + y(t) \end{cases}$ (ii) $\begin{cases} x'(t) = x(t) - y(t) \\ y'(t) = -x(t) + x(t)y(t) \end{cases}$
(iii) $\begin{cases} x'(t) = (\cos^2(t) + 1)x(t) - 2y(t) + \sin^2(t)x(t) \\ y'(t) = -2x(t) + 4y(t) \end{cases}$ (iv) $\begin{cases} x'(t) = y(t) \\ y'(t) = \sin(x(t)) - y(t) \end{cases}$

Solutions

(a) $\boxed{\text{(iv)}}$ The zero vector of $\mathcal{P}_4$ is the zero polynomial $z(x) = 0$. $z'(x) = 0$ so $z(x)$ does not belong to the subset (iv). Since it does not contain the zero vector it cannot be a subspace.

(b) $\boxed{\text{(iv)}}$ Note that the definition of T in (iv) is equivalent to the determinant of the matrix. We know that $\det(A+B) \neq \det(A) + \det(B)$ (take the $A, B = I$ if you would like). Therefore the subset is not closed under addition and is not a subspace.

(c) $\boxed{\text{(i)}}$ This system has an explicit dependence on the t variable in the first equation $x(t) - 2e^t y(t)$ so it is not autonomous. Note that (iii) appears to have a dependence on t in the first equation but

$$(\cos^2(t) + 1)x(t) - 2y(t) + \sin^2(t)x(t) = x(t)\cos^2(t) + x(t) - 2y(t) + \sin^2(t)x(t)$$

$$= x(t)(\cos^2(t) + \sin^2(t)) + x(t) - 2y(t) = 2x(t) - 2y(t)$$

so it does not in fact have an explicit t dependence.

7. (a) Determine the 4×4 matrix P which orthogonally projects a vector in $\mathbb{R}^4$ onto the subspace spanned by $\vec{w} = \begin{bmatrix} -2 \\ 0 \\ 6 \\ -3 \end{bmatrix}$.

   (b) Determine the rank of P and the dimension of the null space (i.e. kernel) of P.

   (c) Find $P^8 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Solutions

(a) We can view this as projection onto a one dimensional subspace. Recall that the projection matrix onto $\vec{w}$ is given by $P = \frac{1}{\vec{w}^T\vec{w}}\vec{w}\vec{w}^T$. Therefore we get:

$$P = \frac{1}{49}\begin{bmatrix} -2 \\ 0 \\ 6 \\ -3 \end{bmatrix}\begin{bmatrix} -2 & 0 & 6 & -3 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 4 & 0 & -12 & 6 \\ 0 & 0 & 0 & 0 \\ -12 & 0 & 36 & -18 \\ 6 & 0 & -18 & 9 \end{bmatrix}$$

(b) P maps from the 4 dimensional space $\mathbb{R}^4$ onto the one dimensional space *span* $\vec{w}$. Therefore the kernel must be dimension 3 and by rank-nullity the rank is $4 - 3 = 1$.

(c) Since P is a projection matrix, $P^2 = P$. As a result $P^8 = P$ and

$$P^8\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = P\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 10 \\ 0 \\ -30 \\ 15 \end{bmatrix}$$

8. Do there exist constants $c_1, c_2, c_3 \in \mathbb{R}$, where at least one of them is non-zero, such that:

$$c_1 \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \\ \cos t \end{bmatrix} = c_3 \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}$$

on some interval of (a, b) where $a < b$? Justify your answer.

Solutions Let

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} \cos t \\ \sin t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_3(t) = \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}.$$

We compute their Wronskian:

$$W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(t) = \begin{vmatrix} \sin t & \cos t & \cos t \\ \cos t & \sin t & 0 \\ \cos t & \cos t & 0 \end{vmatrix} = \cos t \begin{vmatrix} \cos t & \sin t \\ \cos t & \cos t \end{vmatrix} = \cos^2 t(\cos t - \sin t).$$

As $W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(t)$ is not constantly zero (for instance at $t = 0$), the vector functions $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent. Thus the constants c_1, c_2, c_3 must be all equal to zero.

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Math 2400 Final Exam 05/04/2023

1. Let A be the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

   (a) Find a basis for the row space of A, i.e. for the span of the rows of A.

   (b) Find a basis for the image of A, i.e. the column space of A.

   (c) Find a basis for $\ker(A)$.

   (d) Is the vector $\begin{bmatrix} 0 & 3 & 1 & 2 & 0 \end{bmatrix}$ in the row space of A? Prove or disprove.

2 points for rref(A)

2 points for (a)

2 points for (b)

2 points for (c)

2 points for (d) → 1pt correct answer

→ 1pt justification

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Math 2400 Final Exam 05/04/2023

2. For each following questions, answer True or False. **You need to justify your answer**.

   (a) A linear system with fewer unknowns than equations must have either infinitely many solutions or no solutions.

   (b) If a vector u is a linear combination of vectors v and w, and v is a linear combination of vectors p, q and r, then u must be a linear combination of p, q, r and w.

   (c) For any choice of vectors v_1, v_2 and v_3 in $\mathbb{R}^5$, the subset

   $$\{v_1 + 2v_2,\ v_1 - v_2 + 6v_3,\ 4v_2 + v_3,\ v_2 + v_3\}$$

   of $\mathbb{R}^5$ can never be linearly independent.

   (d) There exists an invertible 10×10 matrix that has 92 entries that are precisely equal to 1.

   (e) There exists a 3×3 matrix P such that the linear transformation $T\colon \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ defined by $T(A) = AP - PA$ has rank 9.

2 points for each answer
↳ 1 if correct True/False
↳ 1 if correct justification

3. (a) Find the general solution of the homogeneous linear system: $\begin{cases} x'(t) = 4x(t) - 2y(t) \\ y'(t) = -2x(t) + 4y(t) \end{cases}$

   (b) Determine which figure below represents the trajectories of the solutions of the system. You must justify your answer.

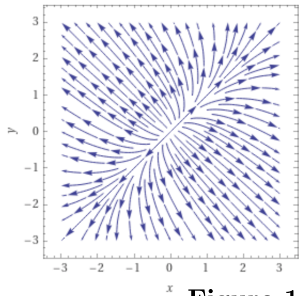

Figure 1

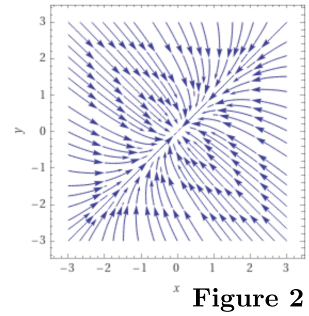

Figure 2

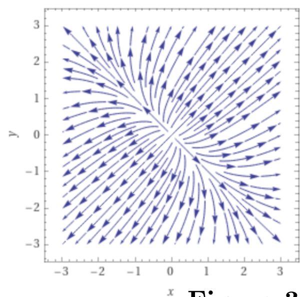

Figure 3

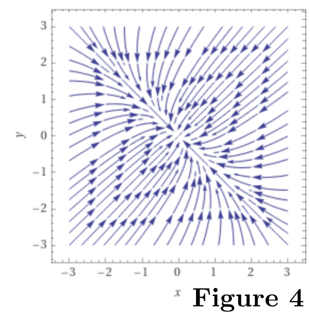

Figure 4

(a) 2 points for eigenvalues
2 points for eigenvectors
1 point for answer

(b) 2 points for figure
3 points for justification

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4. Find the general solution to the ordinary differential equation:

$$f'''(t) - 2f''(t) + f'(t) = 4t + 2e^{2t}.$$

4 points for homogeneous part

↳ 2 for correct auxiliary polynomial

2 for correct roots

1 point for $y_h + y_p$ form

5 points for correct particular sol

↳ 2 for correct guess

↳ 2 for plugging in

↳ 1 for correct constants

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5. Let $\mathcal{M}_{2\times 2}$ be the vector space of 2×2-matrices. Let $T\colon \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be the linear transformation defined as follows:

$$T(A) = \begin{bmatrix} \mathsf{tr}(A) & 0 \\ 0 & \mathsf{tr}(A) \end{bmatrix}$$

where $\mathsf{tr}(A)$ is the trace of a matrix A.

(a) Compute $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$. 1 point

(b) Find the matrix of T with respect to the standard basis of $\mathcal{M}_{2\times 2}$. 2 points

(c) Find a basis for the kernel of T (be sure to express the basis in terms of elements in $\mathcal{M}_{2\times 2}$ and not as column vectors). 2 points for basis + 1 point if in $M_{2\times 2}$

(d) What are the eigenvalues of T? 2 points

(e) Is T diagonalizable? If yes, find a basis in $\mathcal{M}_{2\times 2}$ that diagonalizes T (be sure to express the basis in terms of elements in $\mathcal{M}_{2\times 2}$ and not as column vectors). If not, can you find a Jordan canonical form? 2 points

↳ 1 if diag

↳ 1 if basis in $M_{2\times 2}$

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6. (a) Let $\mathcal{P}_4$ be the vector space of polynomials of degree less than or equal to 4. Among the following subsets of $\mathcal{P}_4$, only one is **not** a subspace, which one is it? You must justify your answer. *You do not need to explain why the others are subspaces.*

(i) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(7) = 0\}$ (ii) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(0) = 0\}$
(iii) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(x) = p(-x)\}$ (iv) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p'(x) \neq 0\}$

(b) Let $\mathcal{M}_2$ be the vector space 2×2-matrices. Among the following functions $T\colon \mathcal{M}_2 \to \mathbb{R}$, only one does **not** define a linear transformation, which one is it? You must justify your answer. *You do not need to explain why the others are linear.*

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + 3b - c + 5d$ (ii) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)^2 + 1 - (a+b+1)^2$

(iii) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$ (iv) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

(c) Among the following systems, only one is **not** autonomous, which one is it? You must justify your answer. *You do not need to explain why the others are autonomous.*

(i) $\begin{cases} x'(t) = x(t) - 2e^t y(t) \\ y'(t) = x^2(t) + y(t) \end{cases}$ (ii) $\begin{cases} x'(t) = x(t) - y(t) \\ y'(t) = -x(t) + x(t)y(t) \end{cases}$

(iii) $\begin{cases} x'(t) = (\cos^2(t) + 1)x(t) - 2y(t) + \sin^2(t)x(t) \\ y'(t) = -2x(t) + 4y(t) \end{cases}$ (iv) $\begin{cases} x'(t) = y(t) \\ y'(t) = \sin(x(t)) - y(t) \end{cases}$

if one is correct → 4 points
if two are corrects → 7 points
if three are correct → 10 points

7. (a) Determine the 4×4 matrix P which orthogonally projects a vector in $\mathbb{R}^4$ onto the subspace spanned by $\vec{w} = \begin{bmatrix} -2 \\ 0 \\ 6 \\ -3 \end{bmatrix}$. 4 points

(b) Determine the rank of P and the dimension of the null space (i.e. kernel) of P. 2 points

(c) Find $P^8 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. 4 points

8. Do there exist constants $c_1, c_2, c_3 \in \mathbb{R}$, where at least one of them is non-zero, such that:

$$c_1 \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \\ \cos t \end{bmatrix} = c_3 \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}$$

on some interval of (a, b) where $a < b$? Justify your answer.

3 for understanding lin indep

2 for seeing Wronskian

3 for computing wronskian correctly

2 for conclusion

MATH 2400 – Final exam – Fall 2022

1. Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$, $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$, and $L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ of $\mathbb{R}^3$ and $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ of $\mathbb{R}^4$.

1. (Continued)

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) What is the rank of L (which is the same as the rank of M_L)? How do you know?

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

2. Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(a) What are the rank and nullity of M?

(b) What are the eigenvalues of M?

(c) Is M diagonalizable? Explain how you know.

3. Let $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

3. (Continued) $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

4. Let $\mathcal{P}_3$ be the vector space of polynomials of degree 3 or less, and consider the inner product on $\mathcal{P}_3$ defined by

$$\langle p\,,\,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

for $p, q \in \mathcal{P}_3$.

Let $\mathcal{S}$ be the subspace of $\mathcal{P}_3$ spanned by $\{1, x\}$. Express the polynomial $z(x) = 20x^3 + 12x^2 + 18x$ as the sum of two polynomials $p_1(x) + p_2(x)$, where $p_1 \in \mathcal{S}$ and p_2 is perpendicular to $\mathcal{S}$.

5. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(a) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$

(b) The origin is the only critical point of the system $\mathbf{x}' = A\mathbf{x}$. What kind of critical point is it?

(c) Find the general solution of the system $\mathbf{x}' = A\mathbf{x} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

6. (a) Find the general solution of the equation $y'' + 2y' + y = 0$.

(b) Now find the general solution of $y'' + 2y' + y = 18e^{2t}$.

(c) Finally, find the general solution of $y'' + 2y' + y = \dfrac{e^{-t}}{t}$

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(a) Of course, the origin is the only critical (equilibrium) point of this system. What kind of critical point is it?

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = (3 - x)(y + 4)$$

$$\frac{dy}{dt} = x(1 - y)$$

MATH 2400 –Final exam – Fall 2022

1. Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that

$$L\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\qquad L\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\qquad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)$, $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$

Since $\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\1\\1\end{bmatrix}$, $L\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$.

Since $\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\\0\end{bmatrix}$.

Since $\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^3$ and $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

The columns of M_L are the images of the (standard) basis vectors and we have two of them. We still need to observe that since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$.

Therefore

$$M_L = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

(c) What is the rank of L (which is the same as the rank of M_L)? How do you know?

The image of L contains of all vectors in $\mathbb{R}^4$ for which $x_1 = x_2$ and $x_3 = x_4$, which is 2-dimensional. (So the rank of L is 2.

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

By the rank-nullity theorem the dimension of the kernel is $3 - 2 = 1$. In fact, it is spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ since all the rows of M_L sum to zero.

2. Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(a) What are the rank and nullity of M?

Since the last two rows of M are multiples of the first row, the rank of M is 1. Therefore the nullity is $3 - 2 = 1$.

(b) What are the eigenvalues of M?

$\lambda = 0$ is an eigenvalue of M with multiplicity 2. The other eigenvalue of M is tr$(M) = 6$ with multiplicity 1.

(c) Is M diagonalizable? Explain how you know.

M is diagonalizable because $\mathbb{R}^3$ has a basis of eigenvectors of M – the two vectors that span the nullspace of M plus the eigenvector for $\lambda = 6$ (which happens to be $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$).

3. Let $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

The eigenvalues of A are $\lambda = 4$ and $\lambda = -1$ (with multiplicity 3).

For $\lambda = 4$, we have $A - 4\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 1 & -1 & 1 & -5 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, we have $A + \mathbf{I} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$, which has rank 2, so there are only two linearly independent eigenvectors, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

We have the eigenvectors for $\lambda = 4$ and $\lambda = -1$, and we need a generalized eigenvector for $\lambda = -1$.

Since $(A + \mathbf{I})^2 = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}_2 = (A + \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and so $P = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

The solution of $\mathbf{y}' = J\mathbf{y}$ is $\mathbf{y} = c_1 e^{4t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c_3 e^{-t}\begin{bmatrix}0\\t\\1\\0\end{bmatrix} + c_4 e^{-t}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$. Or, you could say that $e^{tJ} = \begin{bmatrix} e^{4t} & 0 & 0 & 0 \\ 0 & e^{-t} & te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

The solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = P\mathbf{y} = c_1 e^{4t}\begin{bmatrix}5\\0\\0\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\0\\0\\-1\end{bmatrix} + c_3 e^{-t}\begin{bmatrix}0\\1\\0\\-t\end{bmatrix} + c_4 e^{-t}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$.

4. Let $\mathcal{P}_3$ be the vector space of polynomials of degree 3 or less, and consider the inner product on $\mathcal{P}_3$ defined by

$$\langle p\,,\,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

for $p, q \in \mathcal{P}_3$.

Let $\mathcal{S}$ be the subspace of $\mathcal{P}_3$ spanned by $\{1, x\}$. Express the polynomial $z(x) = 20x^3+12x^2+18x$ as the sum of two polynomials $p_1(x) + p_2(x)$, where $p_1 \in \mathcal{S}$ and p_2 is perpendicular to $\mathcal{S}$.

First we calculate $\langle 1\,,\,x\rangle = \int_{-1}^{1} x\,dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0$, so that $\{1.x\}$ is an orthogonal basis for $\mathcal{S}$. The polynomial p_1 should be the projection of z onto $\mathcal{S}$:

$$\begin{aligned}
p_1 = \operatorname{proj}_{\mathcal{S}} z &= \left(\frac{\langle 1\,,\,z\rangle}{\langle 1\,,\,1\rangle}\right) 1 + \left(\frac{\langle x\,,\,z\rangle}{\langle x\,,\,x\rangle}\right) x \\
&= \left(\frac{\int_{-1}^{1} 20x^3 + 12x^2 + 18x\,dx}{\int_{-1}^{1} 1^2\,dx}\right) 1 + \left(\frac{\int_{-1}^{1} 20x^4 + 12x^3 + 18x^2\,dx}{\int_{-1}^{1} x^2\,dx}\right) x \\
&= \left(\frac{(5x^4 + 4x^3 + 9x^2)\big|_{-1}^{1}}{x\big|_{-1}^{1}}\right) 1 + \left(\frac{(4x^5 + 3x^4 + 6x^3)\big|_{-1}^{1}}{\frac{1}{3}x^3\big|_{-1}^{1}}\right) x \\
&= \frac{8}{2}1 + \frac{20}{\frac{2}{3}}x = 4 + 30x
\end{aligned}$$

And then

$$p_2 = z - p_1 = (20x^3 + 12x^2 + 18x) - (30x + 4) = 20x^3 + 12x^2 - 12x - 4$$

Of course $z = p_1 + p_2$, and we can check that

$$\langle p_2\,,\,1\rangle = \int_{-1}^{1} 20x^3 + 12x^2 - 12x - 4\,dx = 0 + 8 - 0 - 8 = 0$$

and

$$\langle p_2\,,\,x\rangle = \int_{-1}^{1} 20x^4 + 12x^3 - 12x^2 - 4x\,dx = 8 + 0 - 8 - 0 = 0$$

so that p_2 really is orthogonal to $\mathcal{S}$.

5. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(a) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$

The characteristic polynomial of A is $(-\lambda)(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.
For $\lambda = 2$, $A - 2\mathbf{I} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
For $\lambda = -1$, $A + \mathbf{I} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
The general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) The origin is the only critical point of the system $\mathbf{x}' = A\mathbf{x}$. What kind of critical point is it?

Since the eigenvalues of A are real and have opposite signs, the origin is a saddle point.

(c) Find the general solution of the system $\mathbf{x}' = A\mathbf{x} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Undetermined coefficients: Guess $\mathbf{x}_p = e^t \mathbf{a}$. Then $e^t \mathbf{a} = e^t A\mathbf{a} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so we need $\mathbf{a}$ to satisfy $(\mathbf{I} - A)\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The linear system $\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -1 & 0 & 1 \end{array}\right]$ has solution $\mathbf{a} = \begin{bmatrix} -1 \\ -3/2 \end{bmatrix}$, so $\mathbf{x}_p = -\frac{1}{2} e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$$\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \frac{1}{2} e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Variation of parameters: From (a) we have a solution matrix for the homogeneous system: $X = \begin{bmatrix} e^{2t} & -2e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}$, with $\det(X) = 3e^t$ and $X^{-1} = \frac{1}{3} e^{-t} \begin{bmatrix} e^{-t} & 2e^{-t} \\ -e^{2t} & e^{2t} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-2t} & 2e^{-2t} \\ -e^t & e^t \end{bmatrix}$. So we have

$$\mathbf{x}_p = X \int X^{-1} \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} = X \int \frac{1}{3} \begin{bmatrix} 4e^{-t} \\ -e^{2t} \end{bmatrix} = X \begin{bmatrix} -\frac{4}{3} e^{-t} \\ -\frac{1}{6} e^{2t} \end{bmatrix} = \begin{bmatrix} -e^t \\ -\frac{3}{2} e^t \end{bmatrix}$$

which agrees with the undetermined coefficients version.

6. (a) Find the general solution of the equation $y'' + 2y' + y = 0$.

The roots of the auxiliary equation are -1 and -1 so the solution is $y = c_1 e^{-t} + c_2 t e^{-t}$

(b) Now find the general solution of $y'' + 2y' + y = 18e^{2t}$.

Undetermined coefficients: Guess $y_p = Ae^{2t}$, then $y_p' = 2Ae^{2t}$ and $y_p'' = 4Ae^{2t}$, therefore $y_p'' + 2y_p' + y_p = 9Ae^{2t}$ so $A = 2$. Conclude

$$y = 2e^{2t} + c_1 e^{-t} + c_2 t e^{-t}$$

(c) Finally, find the general solution of $y'' + 2y' + y = \dfrac{e^{-t}}{t}$

Variation of parameters: we have $W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (-t+1)e^{-t} \end{vmatrix} = e^{-2t}$, so the particular solution is

$$y_p = e^{-t} \int \frac{-te^{-t}\, e^{-t}}{te^{-2t}}\, dt + te^{-t} \int \frac{e^{-t}\, e^{-t}}{te^{-2t}}\, dt = -te^{-t} + te^{-t} \ln t$$

Therefore the general solution is

$$y = -te^{-t} + te^{-t} \ln t + c_1 e^{-t} + c_2 t e^{-t}$$

Note, you could absorb the first term into the c_2 term.

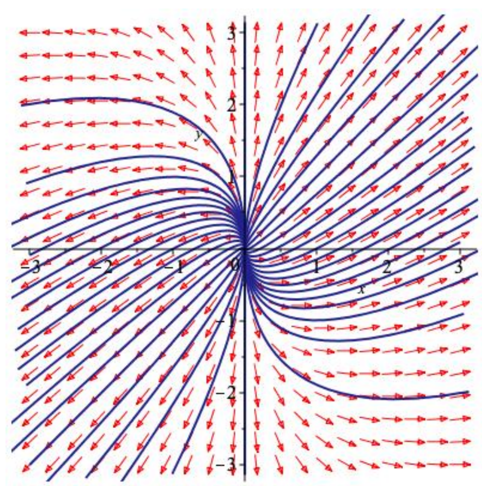
7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since the matrix A is lower triangular, its eigenvalues are $\lambda = 1$ and $\lambda = 2$. Therefore the origin is an unstable node (improper source node).

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

We need the eigenvectors of A. For $\lambda = 1$, an eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and for $\lambda = 2$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The trajectories will emanate from the origin, tangent to the y-axis (i.e., in the direction of the eigenvector for the smaller eigenvalue, $\lambda = 1$) and curve so they become parallel to the line $y = x$ (parallel to the eigenvector for the larger eigenvalue) as $t \to \infty$. The straight-line trajectories are the positive and negative y-axis and the parts of the line $y = x$ in the first and third quadrants.



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = (3 - x)(y + 4)$$

$$\frac{dy}{dt} = x(1 - y)$$

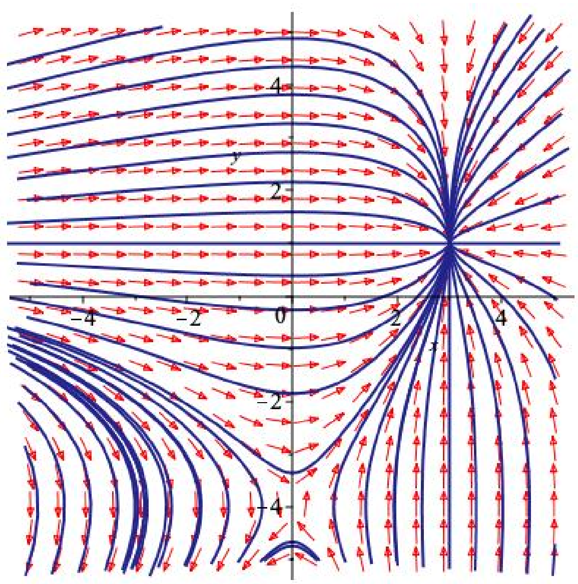
For the critical points, we need both right-hand sides to be zero. The first one is zero if $x = 3$ or $y = -4$ and the second if $x = 0$ or $y = 1$. Therefore there are two critical points $(3, 1)$ and $(0, -4)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} -y-4 & 3-x \\ 1-y & -x \end{bmatrix}$.

At $(3, 1)$ we have $J(3, 1) = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}$, which is a stable node (or improper sink).

At $(0, -4)$, $J(0, -4) = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}$, which is a saddle.

For the record:



Final Exam, Math 240: Calculus III

May 9, 2022

No book, paper or electronic device may be used, other than a hand-written note sheet at most $8.5'' \times 11''$ in size. Cell phones should be **in your bags** and **turned off**.

This examination consists of one (1) short-answer question, one (1) True/False question, and six (6) long answer questions. Partial credits will be given only for long-answer questions, and only when a substantial part of the problem has been worked out. Please **show all your work** for these questions. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please **box your answers** for **all** questions.

Name (printed):

Instructor:

TA:

Recitation Time:

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

____

Your signature

1	2	3	4	5	6	7	8	Total

1. (Short answer question) Let V be a vector space over the real numbers, Let $\mathrm{I}_V : V \to V$ be the identity map on V, let $T_0 : V \to V$ be the zero transformation from V to V, and let

$$P := \mathrm{proj}_W : \mathbb{R}^4 \to \mathbb{R}^4$$

be the orthogonal projection from $\mathbb{R}^4$ to the vector subspace W of $\mathbb{R}^4$, where W consisting of all vectors $[x_1, x_2, x_3, x_4]^T$ in $\mathbb{R}^4$ such that $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$.

Describe explicitly the vector spaces in questions (a)–(d) below. (No justification is necessary. Please **box your answers**.)

(a) What is $\mathrm{Ker}(\mathrm{I}_V)$?

$= \{0\}$ (0-dimensional)

(b) What is $\mathrm{Image}(\mathrm{I}_V)$?

$= V$ $\mathrm{I}_V(v) = v$.

(c) What is $\mathrm{Ker}(T_0)$?

$= V$

$x + y + z = 0$ in $\mathbb{R}^3$

$$P := \mathrm{proj}_W : \mathbb{R}^4 \to \mathbb{R}^4$$

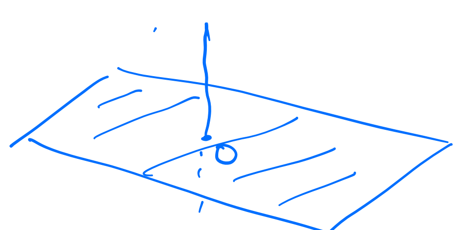
$\mathbb{R}^4$ to the vector subspace W of $\mathbb{R}^4$
hat $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$.

(d) What is $\mathrm{Ker}(P)$?

1-dimensional with basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$

3 dimensional

$\mathbb{R}^4$



If A were $m \times n$ then A surjective if $\dim \mathrm{Im}(A) = \mathrm{rank}(A) = m$

$A: \mathbb{R}^n \to \mathbb{R}^m$

$m[\,] = m[\;\;]\,[\,]$

2. Let

$$A_k = \begin{bmatrix} 5 & 17 & k & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix},$$

← 1
← independent.
← 5

$5 \cdot \text{Row}4 + 1 \cdot \text{Row}2 = [5 \quad 17 \quad 1 \quad 0]$

and consider the linear transformations

$$L_{A_k} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

given by left multiplication by A_k, where $k \in \mathbb{R}$ is a parameter.

$\dim \text{Im} L_{A_k} = 4$

injective (1-1)

(a) Determine the all values of the parameter $k \in \mathbb{R}$ such that L_{A_k} is surjective (onto).

Need A_k to be invertible $\iff$ rank $A_k = 4$

$\iff \det A_k \neq 0$

If $k = 1$ A_k is not invertible (rank 3)

$k \neq 1$ A_k is invertible

$$\det A_k = -1 \begin{vmatrix} 5 & 17 & k \\ 0 & 7 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -1\,(17 - 10 - 7k)$$

$$= 7k - 7 \neq 0 \iff k \neq 1$$

(b) Find a basis for the images of L_{A_k} for all k such that the map L_{A_k} is **not** onto.

$$\begin{bmatrix} 5 & 17 & 1 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-2 \quad 1 \quad -7$

column space basis $\left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$A^{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$

injective : 1-1 : $\ker A = \{0\} \iff$ nullity$(A) = 0$ | impossible if $n > m$

surjective : onto : $\text{Im} A = \mathbb{R}^m \iff$ rank$(A) = m$ | impossible if $n < m$

bijective : invertible : both of the above | impossible unless $n = m$.

3. Let $M_{2\times 2}$ be the real vector space of 2×2-matrices with real coefficients. Define a linear transformation $L : M_{2\times 2} \to M_{2\times 2}$ as follows:

$$L : M_{2\times 2} \longrightarrow M_{2\times 2}$$
$$L : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a+3b & 3a+b \\ 2c+6d & 3d \end{bmatrix}$$

Find a basis in $M_{2\times 2}$ that diagonalizes L. Please make sure to justify your steps.

P

$PDP^{-1} = L$

$$\begin{bmatrix} a+3b \\ 3a+b \\ 2c+6d \\ 3d \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

L

$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

$\lambda = 4, -2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ $\lambda = 4$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda = -2$ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

$\lambda = 2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\lambda = 3$ $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$\Rightarrow$ $\lambda = 2$ $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\lambda = 3$ $\begin{bmatrix} 0 \\ 0 \\ 6 \\ 1 \end{bmatrix}$

Basis that diagonalizes L is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 6 & 1 \end{bmatrix} \right\}$

4. Let $A \in M_{3\times 3}$ be a symmetric 3×3 matrix with coefficients in $\mathbb{R}$ such that $A^2 = 6A$.

(a) Suppose that λ is an eigenvalue of A. Show that either $\lambda = 0$ or $\lambda = 6$.

$v \neq 0$

$Av = \lambda v$

$A^2 v = \lambda A v$

$6Av = \lambda A v$

$0 = (\lambda - 6) Av$

Either $\lambda = 6$ or $Av = 0$

$\Rightarrow Av = 0v$

i.e. 0 is an eigenvalue of A

$Av = 6v$

$A = RDR^T$ $\quad A^2 = 6A$

$A^2 = RD^2R^T$ $\quad \Rightarrow D^2 = 6D$

$\begin{bmatrix} d_1^2 & & \\ & d_2^2 & \\ & & d_3^2 \end{bmatrix} = \begin{bmatrix} 6d_1 & & \\ & 6d_2 & \\ & & 6d_3 \end{bmatrix}$

$d^2 = 6d$

$d = 0, 6$

$A = PDP^{-1}$

(b) Suppose that the column space (or column span) of A is spanned by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = 6$: v_1, v_2; $\lambda = 0$: $v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Determine the matrix A.

(You can use the statement (a). Note that the two vectors above are perpendicular to each other.)

A has rank 2 $\Rightarrow$ has an eigenvector for $\lambda = 0$

has two l.i. no. eigenvectors for $\lambda = 6$

$D = \begin{bmatrix} 6 & & \\ & 6 & \\ & & 0 \end{bmatrix}$

Spectral decomposition:

For a symmetric matrix A: with λ_i, v_i

$$A = \frac{\lambda_1}{\langle v_1, v_1\rangle} v_1 v_1^T + \frac{\lambda_2}{\langle v_2, v_2\rangle} v_2 v_2^T + \cdots$$

$$= \frac{6}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{6}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{0}{3} v_3 v_3^T$$

$$= \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

$\langle v_1, v_1 \rangle = v_1^T v_1$

$[1\ 1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [3]$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[1\ 1\ 1] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

3×1 1×3

v_1 v_2 in image of A.

$Aw_1 = v_1$ $\quad A^2 w_1 = Av_1 \Rightarrow 6Av_1 = Av_1$

$Aw_2 = v_2$ $\quad 6v_1 = Av_1$

$D = \begin{bmatrix} 6 & & \\ & 6 & \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6a \\ 6b \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$

5. Consider the following non-linear autonomous system

$$\frac{dx}{dt} = y(x-1) \qquad = 0$$
$$\frac{dy}{dt} = -9\sin(\pi x) - y \qquad = 0$$

(a) Find all *critical points* (also called *equilibrium points*) of this autonomous system. List all of them if there are only finitely many, and use suitable parameters to describe all equilibrium points if there are infinitely many.

$y(x-1)=0 \Rightarrow$ Either $y=0$ or $x=1 \to y=0$ $(1,0)$

$\sin(\pi x)=0$

$\Rightarrow x$ is an integer

Critical points: $\{(n,0)$ where n is an integer$\}$

det $J < 0$

(b) Among the critical points, which ones are saddles? List all of them if there are only finitely many, and use suitable parameters to describe all equilibrium points which are saddles if there are infinitely many.

$$\frac{dx}{dt} = y(x-1) \qquad = 1$$
$$\frac{dy}{dt} = -9\sin(\pi x) - y$$

$$J = \begin{bmatrix} y & x-1 \\ -9\pi\cos\pi x & -1 \end{bmatrix} \qquad \cos n\pi = (-1)^n$$

$$J(n,0) = \begin{bmatrix} 0 & n-1 \\ 9\pi(-1)^{n+1} & -1 \end{bmatrix} \qquad \det J(n,0) = 9\pi(n-1)(-1)^n < 0$$

When is $(n-1)(-1)^n < 0$?

Either when $(n-1) > 0$ and $(-1)^n < 0 \Rightarrow$ $n>1$ and n is odd

or when $(n-1) < 0$ and $(-1)^n > 0 \Rightarrow$ $n<1$ and n is even

$\ldots (-4,0)\ (-2,0)\ (0,0)\ \&\ (3,0)\ (5,0)\ (7,0) \ldots$

(c) Among the critical points P, which ones have the property that for every solution $(x(t),y(t))$ of the autonomous system whose initial point $(x(0),y(0))$ is sufficiently close to to P, we have $\lim_{t\to\infty}(x(t),y(t)) = P$? List all of them if there are only finitely many, and use suitable parameters to describe all stable nodes if there are infinitely many.

asymptotically stable — both eigenvalues must have negative real parts

Eigenvalues: $-\lambda(-1-\lambda) - (n-1)9\pi(-1)^{n+1}$

$\lambda^2 + \lambda + (-1)^n 9\pi(n-1)$

$$\lambda = \frac{-1 \pm \sqrt{1 + (-1)^{n+1} 36\pi(n-1)}}{2}$$

Four cases:

$n>1$ even $\Rightarrow \sqrt{\#}$ negative. $-\frac{1}{2} \pm i\sqrt{\#} \Leftarrow$ asymp. stable.

$n>1$ odd $\Rightarrow \sqrt{\#}$ big positive $-\frac{1}{2} + \sqrt{\#} > 0$ – saddles

$n<1$ even $\Rightarrow \sqrt{\#}$ big positive saddles

$n<1$ odd $\Rightarrow -\frac{1}{2} \pm i\sqrt{\#} \Leftarrow$ stable.

Only remaining case $n=1$ eigenvalues are $-1, 0$. not a.s.

6. (a) Find the general solution of the differential equation

$$\left(D^2+4\right)^2 y(t) = 0,$$

$y'''' + 8y'' + 16y = 0$

where $D = \frac{d}{dt}$.

$(r^2+4)^2 = 0 \quad r = \pm 2i$ each w/ multiplicity 2.

$y = c_1 e^{2it} + c_2 t e^{2it} + c_3 e^{-2it} + c_4 t e^{-2it}$

$= a_1 \cos 2t + a_2 t\cos 2t + a_3 \sin 2t + a_4 t \sin 2t$.

(b) Find a solution of the differential equation

$$\left(D^2+4\right)^2 y(t) = \cos(2t)$$

$\frac{1}{2}e^{2it} + \frac{1}{2}e^{-2it}$

$y_p = A t^2 \cos 2t + B t^2 \sin 2t$

or

$y_p = C t^2 e^{2it} + D t^2 e^{-2it}$

$y'''' + 8y'' + 16y = \frac{1}{2}e^{2it}$

16 $\quad y_p = Ct^2 e^{2it}$

$y_p' = 2iCt^2 e^{2it} + 2Cte^{2it}$

8 $\quad y_p'' = -4Ct^2e^{2it} + 8iCte^{2it} + 2Ce^{2it}$

$y_p''' = -8iCt^2e^{2it} - 24Cte^{2it} + 12iCe^{2it}$

1 $\quad y_p'''' = 16Ct^2e^{2it} - 64iCte^{2it} - 48Ce^{2it}$

$y_p'''' + 8y_p'' + 16y_p = -32Ce^{2it} = \frac{1}{2}e^{2it} \Rightarrow C = -\frac{1}{64}$.

$-48C \quad 16C$

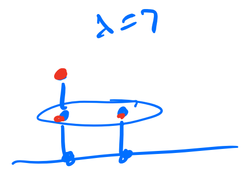
$y_p = -\frac{1}{64}t^2e^{2it} - \frac{1}{64}t^2e^{-2it}$

$= -\frac{1}{32}t^2 \frac{e^{2it}+e^{-2it}}{2}$

$= -\frac{1}{32}t^2\cos 2t$

7. Let A be a real matrix with characteristic polynomial $(\lambda - 7)^5$. You are told that:

- $\operatorname{rank}(A - 7\,I_5) = 3$,
- $\operatorname{rank}((A - 7\,I_5)^2) = 1$,
- $\operatorname{rank}((A - 7\,I_5)^3) = 0$.



(a) Find a Jordan canonical form J associated to A. (Note that J is only unique up to a permutation of the Jordan blocks.)

(b) Find a general solution of the homogeneous system $\mathbf{x}'(t) = J\,\mathbf{x}(t)$, where J is the Jordan form of A that you found, and $\mathbf{x}(t)$ is a 5-vector of smooth functions.

8. For each of the following statements (a)–(c), decide whether it is True or False, and **circle your answers**. No justification is necessary.

(a) Let n be a positive integers. Let $\mathrm{M}_{n\times n}(\mathbb{R})$ be the vector space over the real numbers $\mathbb{R}$ consisting of all $n \times n$ matrices with entries in $\mathbb{R}$. The set

$$S_1 := \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) \mid \operatorname{Trace}(A) = 1\}$$

is a vector subspace of $\mathrm{M}_{n\times n}(\mathbb{R})$.

Answer (circle one): True **(False)**

0 matrix not in S

(b) The empty set $S_2 := \emptyset$ is a vector subspace of every vector space.

Answer (circle one): True **(False)**

(c) Let n be a positive integer. The set

$$S_3 := \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) \mid \ A = A^T\}$$

$B = B^T$

is a vector subspace of $\mathrm{M}_{n\times n}(\mathbb{R})$.

Answer (circle one): **(True)** False

$(A+B)^T = A^T + B^T = A + B$

$(kA)^T = k A^T = kA$

Final Exam, Math 240: Calculus III

May 9, 2022

No book, paper or electronic device may be used, other than a hand-written note sheet at most $8.5'' \times 11''$ in size. Cell phones should be **in your bags** and **turned off**.

This examination consists of one (1) short-answer question, one (1) True/False question, and six (6) long answer questions. Partial credits will be given only for long-answer questions, and only when a substantial part of the problem has been worked out. Please **show all your work** for these questions. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please **box your answers** for **all** questions.

Name (printed):

Instructor:

TA:

Recitation Time:

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

____

Your signature

1	2	3	4	5	6	7	8	Total

1. (Short answer question) Let V be a vector space over the real numbers, Let $\mathrm{I}_V : V \to V$ be the identity map on V, let $T_0 : V \to V$ be the zero transformation from V to V, and let

$$P := \mathrm{proj}_W : \mathbb{R}^4 \to \mathbb{R}^4$$

be the orthogonal projection from $\mathbb{R}^4$ to the vector subspace W of $\mathbb{R}^4$, where W consisting of all vectors $[x_1, x_2, x_3, x_4]^T$ in $\mathbb{R}^4$ such that $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$.

Describe explicitly the vector spaces in questions (a)–(d) below. (No justification is necessary. Please **box your answers**.)

(a) What is $\mathrm{Ker}(\mathrm{I}_V)$?

(b) What is $\mathrm{Image}(\mathrm{I}_V)$?

(c) What is $\mathrm{Ker}(T_0)$?

(d) What is $\mathrm{Ker}(P)$?

2. Let

$$A_k = \begin{bmatrix} 5 & 17 & k & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix},$$

and consider the linear transformations

$$L_{A_k} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4,$$

given by left multiplication by A_k, where $k \in \mathbb{R}$ is a parameter.

(a) Determine the all values of the parameter $k \in \mathbb{R}$ such that L_{A_k} is surjective (onto).

(b) Find a basis for the images of L_{A_k} for all k such that the map L_{A_k} is **not** onto.

3. Let $M_{2\times 2}$ be the real vector space of 2×2-matrices with real coefficients. Define a linear transformation $L : M_{2\times 2} \to M_{2\times 2}$ as follows:

$$L : M_{2\times 2} \longrightarrow M_{2\times 2}$$
$$L : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a + 3b & 3a + b \\ 2c + 6d & 3d \end{bmatrix}$$

Find a basis in $M_{2\times 2}$ that diagonalizes L. Please make sure to justify your steps.

4. Let $A \in \mathrm{M}_{3\times 3}$ be a symmetric 3×3 matrix with coefficients in $\mathbb{R}$ such that $A^2 = 6A$.

(a) Suppose that λ is an eigenvalue of A. Show that either $\lambda = 0$ or $\lambda = 6$.

(b) Suppose that the column space (or column span) of A is spanned by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Determine the matrix A.

(You can use the statement (a). Note that the two vectors above are perpendicular to each other.)

5. Consider the following non-linear autonomous system

$$\frac{dx}{dt} = y(x-1)$$
$$\frac{dy}{dt} = -9\sin(\pi x) - y$$

(a) Find all *critical points* (also called *equilibrium points*) of this autonomous system. List all of them if there are only finitely many, and use suitable parameters to describe all equilibrium points if there are infinitely many.

(b) Among the critical points, which ones are saddles? List all of them if there are only finitely many, and use suitable parameters to describe all equilibrium points which are saddles if there are infinitely many.

(c) Among the critical points P, which ones have the property that for every solution $(x(t), y(t))$ of the autonomous system whose initial point $(x(0), y(0))$ is sufficiently close to to P, we have $\lim_{t\to\infty}(x(t), y(t)) = P$? List all of them if there are only finitely many, and use suitable parameters to describe all stable nodes if there are infinitely many.

6. (a) Find the general solution of the differential equation

$$\left(D^2+4\right)^2 y(t)=0,$$

where $D=\frac{d}{dt}$.

(b) Find a solution of the differential equation

$$\left(D^2+4\right)^2 y(t)=\cos(2t)$$

7. Let A be a real matrix with characteristic polynomial $(\lambda - 7)^5$. You are told that:

- $\mathsf{rank}\big(A - 7\,\mathrm{I}_5\big) = 3$,
- $\mathsf{rank}\big((A - 7\,\mathrm{I}_5)^2\big) = 1$,
- $\mathsf{rank}\big((A - 7\,\mathrm{I}_5)^3\big) = 0$.

(a) Find a Jordan canonical form J associated to A. (Note that J is only unique up to a permutation of the Jordan blocks.)

(b) Find a general solution of the homogeneous system $\mathbf{x}'(t) = J\,\mathbf{x}(t)$, where J is the Jordan form of A that you found, and $\mathbf{x}(t)$ is a 5-vector of smooth functions.

8. For each of the following statements (a)–(c), decide whether it is True or False, and **circle your answers**. No justification is necessary.

(a) Let n be positive integers. Let $\mathrm{M}_{n\times n}(\mathbb{R})$ be the vector space over the real numbers $\mathbb{R}$ consisting of all $m \times n$ matrices with entries in $\mathbb{R}$. The set

$$S_1 := \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) | \ \mathrm{Trace}(A) = 1\}$$

is a vector subspace of $\mathrm{M}_{m\times n}(\mathbb{R})$.

Answer (circle one): True False

(b) The empty set $S_2 := \emptyset$ is a vector subspace of every vector space.

Answer (circle one): True False

(c) Let n be a positive integer. The set

$$S_3 := \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) | \ \ A = A^T\}$$

is a vector subspace of $\mathrm{M}_{n\times n}(\mathbb{R})$.

Answer (circle one): True False

Math 240 Final Exam

Fall 2021

1. Let $B = \{1, x, x^2\}$ be the standard basis for the vector space $\mathcal{P}_2$ of polynomials of degree at most 2 with real coefficients, and let $C = \{x(x+1)\ , x(x-1)\ , (x-1)(x-2)\}$ be another basis for $\mathcal{P}_2$. Find the matrix $P_{C\leftarrow B}$, i.e., the change of basis matrix which satisfies

$$[z]_C = P_{C\leftarrow B}[z]_B$$

for any $z \in \mathcal{P}_2$. As usual, by $[z]_C$ we mean the element z expressed in C coordinates.

2. Let A be a 4×4 matrix of integers. Assume that A has two eigenvalues; one λ_1 with algebraic multiplicity 1 and another λ_2 with algebraic multiplicity 3 (so the characteristic polynomial of A factors as $(\lambda - \lambda_1)(\lambda - \lambda_2)^3$). Say we know that the dimension $\dim(\ker(A - \lambda_1 I)) = 1$ and $\dim(\ker(A - \lambda_2 I)) = 1$.

(a) Find the Jordan canonical form J for A in terms of λ_1 and λ_2.

(b) With A as above, assume further that the determinant and trace of A are $\det(A) = 8$, $\mathrm{tr}(A) = 7$. Find λ_1 and λ_2.

3. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

(a) Find the Jordan canonical form J of A.

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogenous system $\mathbf{x}' = A\mathbf{x}$.

4. Suppose that $y(t)$ is the height above the ground at time t of a miniature helicopter carried underneath the carriage of a Martian rover. When the rover touches down at time $t = 0$, one has $y(0) = .4$ meters. The rover bounces when it lands. This leads to $y'(0) = -1$ meters per second and the differential equation

$$y''(t) = -2(y(t) - .4) - 2y'(t).$$

The helicopter will be damaged if $y(t)$ ever becomes negative for some $t > 0$. Will this happen? Justify your answer.

Hints: Minimize the function $y(t)$. Remember that $y'(t) = 0$ when $y(t)$ is minimal. You can use the fact that

$$\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2} < 0.4 \cdot e^{\pi/4}.$$

5. Consider systems of the form $x' = Ax$ where A is a matrix of the form

$$\begin{bmatrix} 2 & 1 \\ a^2 - 1 & 0 \end{bmatrix}$$

in which a can be any real number.

(a) For what values of a, if any, do all trajectories flow towards the origin (i.e., $\lim_{t \to \infty} x(t) = (0, 0)$ for all trajectories, a.k.a. solutions to they system, x)?

(b) For what values of a, if any, do all trajectories flow out of the origin (i.e., $\lim_{t \to -\infty} x(t) = (0, 0)$ for all trajectories, a.k.a. solutions to they system, x)?

(c) For what values of a, if any, do some trajectories flow out of the origin and all other trajectories flow towards the origin?

6. Consider a system

$$x' = 2x(x-2)(y+3)$$
$$y' = (x+2)(y+4)$$

Find all critical points and describe the behavior of the system near these critical points.

7. The Texas state legislature decides there are too many medical procedures going on in their state that they don't approve of. They decide to impose some new conditions on which procedures can be done based on the following three variables:

- The time t in weeks that a person has needed the procedure.
- The net wealth w in hundreds of thousands of dollars of the person seeking the procedure.
- The contributions d in thousands of dollars the person has made to members of the legislature in the past year.

The legislature passes a law saying that the vector $v = (t, w, d)$ corresponding to the person requesting the procedure must be at distance at most 10 from the subspace W of $\mathbb{R}^3$ spanned by the vectors $w_1 = (0, 1, 1)$ and $w_2 = (0.1, 1, 0)$. If $v = (10, 8, 10)$, will the procedure be allowed?

8. Suppose $y(t)$ is the displacement of a door to the U.S. Capitol at time t. As a result of the strength of the door and rioters pushing on it, $y(t)$ satisfies the differential equation

$$y''(t) = -y(t) + f(t)$$

where $f(t)$ results from the force applied by the rioters. A quick-thinking Capitol police officer realizes three things:

- $y(0) = 0$ and $y'(0) = 1$.
- The door will break when $y(t) = 1$.
- Reinforcements won't arrive till $t = 1$. For $0 \leqslant t \leqslant 1$ the rioters will maintain $f(t) = 1$. It's not clear what $f(t)$ will be for $t \geqslant 1$.

Fortunately, the officer took Math 2400. From the above information, can she decide whether the door will break before $t = 1$? If so, will it break or not before $t = 1$? Justify your answers.

Math 240 Final Exam

Fall 2021

1. Let $B = \{1, x, x^2\}$ be the standard basis for the vector space $\mathcal{P}_2$ of polynomials of degree at most 2 with real coefficients, and let $C = \{x(x+1)\ , x(x-1)\ , (x-1)(x-2)\}$ be another basis for $\mathcal{P}_2$. Find the matrix $P_{C\leftarrow B}$, i.e., the change of basis matrix which satisfies

$$[z]_C = P_{C\leftarrow B}[z]_B$$

for any $z \in \mathcal{P}_2$. As usual, by $[z]_C$ we mean the element z expressed in C coordinates.

To begin, we note that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_B$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_C = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}_B$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_C = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_B$.
This means that

$$P_{B\leftarrow C} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

and $P_{C\leftarrow B} = P_{B\leftarrow C}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -4 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

Thus $P_{C\leftarrow B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$.

(So for instance

$$\begin{aligned} 1 &= \frac{1}{2}x(x+1) - x(x-1) + \frac{1}{2}(x-1)(x-2) = \frac{1}{2}x^2 + \frac{1}{2}x - x^2 + x + \frac{1}{2}x^2 - \frac{3}{2}x + 1 \\ x &= \frac{1}{2}x(x+1) - \frac{1}{2}x(x-1) = \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{2}x \\ x^2 &= \frac{1}{2}x(x+1) + \frac{1}{2}x(x-1) = \frac{1}{2}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x \end{aligned}$$

)

2. Let A be a 4×4 matrix of integers. Assume that A has two eigenvalues; one λ_1 with algebraic multiplicity 1 and another λ_2 with algebraic multiplicity 3 (so the characteristic polynomial of A factors as $(\lambda - \lambda_1)(\lambda - \lambda_2)^3$). Say we know that the dimension $\dim(\ker(A - \lambda_1 I)) = 1$ and $\dim(\ker(A - \lambda_2 I)) = 1$.

(a) Find the Jordan canonical form J for A in terms of λ_1 and λ_2.

Because $\dim(\ker(A - \lambda_2 I)) = 1$, there is only one Jordan block with eigenvalue λ_2. Therefore

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

(b) With A as above, assume further that the determinant and trace of A are $\det(A) = 8$, $\operatorname{tr}(A) = 7$. Find λ_1 and λ_2.

We need $\lambda_1 + 3\lambda_2 = 7$ and $\lambda_1 \lambda_2^3 = 8$. It would seem that $\lambda_1 = 1$ and $\lambda_2 = 2$.

3. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

(a) Find the Jordan canonical form J of A.

Since A is lower triangular, the characteristic polynomial is $(3-\lambda)(1-\lambda)^2$ so the eigenvalues are $\lambda = 3$ and $\lambda = 1$. And since $A - \mathbf{I} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ has rank 2 (and nullity 1), the Jordan form of A is

$$J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

Since $A - 3\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{bmatrix}$, the eigenvector for $\lambda = 3$ is $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.
From $A - \mathbf{I}$ computed in part (a) we see that the only linearly independent eigenvector for $\lambda = 2$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then, $(A-\mathbf{I})^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$, which of course has rank 1, and its kernel is the span of $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. So we can take $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as the first generalized eigenvector in the chain and then $(A - \mathbf{I}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So the matrix $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

$$e^{tJ} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogenous system $\mathbf{x}' = A\mathbf{x}$.

Obtain the general solution of $\mathbf{x}' = A\mathbf{x}$ from the columns of $Pe^{tJ} = \begin{bmatrix} 2e^{3t} & 0 & 0 \\ 0 & 0 & e^t \\ -e^{3t} & e^t & te^t \end{bmatrix}$:

$$\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix}$$

4. Suppose that $y(t)$ is the height above the ground at time t of a miniature helicopter carried underneath the carriage of a Martian rover. When the rover touches down at time $t = 0$, one has $y(0) = .4$ meters. The rover bounces when it lands. This leads to $y'(0) = -1$ meters per second and the differential equation

$$y''(t) = -2(y(t) - .4) - 2y'(t).$$

The helicopter will be damaged if $y(t)$ ever becomes negative for some $t > 0$. Will this happen? Justify your answer.

Hints: Minimize the function $y(t)$. Remember that $y'(t) = 0$ when $y(t)$ is minimal. You can use the fact that

$$\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2} < 0.4 \cdot e^{\pi/4}.$$

In standard form, the differential equation is $y'' + 2y' + 2y = 0.8$. The auxiliary equation is $r^2 + 2r + 2 = 0$, with solution $r = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. So the complementary solution is $y_c = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$ and (by undetermined coefficients) the particular solution is $y_p = 0.4$. The general solution is thus

$$y = 0.4 + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

For $y(0) = 0.4$, we'll need $c_1 = 0$. Then

$$y' = c_2 e^{-t}(\cos t - \sin t)$$

and $y'(0) = -1$ means $c_2 = -1$. Thus $y(t) = 0.4 - e^{-t}\sin t$. From the expression for y' above, we'll have $y' = 0$ when $\cos t = \sin t$, so for instance when $t = \pi/4$. And $y(\pi/4) = 0.4 - e^{-\pi/4}\sqrt{2}/2$. By the hint, this quantity is bigger than zero, so the helicopter will **not** be damaged.

5. Consider systems of the form $x' = Ax$ where A is a matrix of the form

$$\begin{bmatrix} 2 & 1 \\ a^2 - 1 & 0 \end{bmatrix}$$

in which a can be any real number.

(a) For what values of a, if any, do all trajectories flow towards the origin (i.e., $\lim_{t\to\infty} x(t) = (0,0)$ for all trajectories, a.k.a. solutions to they system, x)?

The eigenvalues of A are the roots of $(2-\lambda)(-\lambda) - (a^2 - 1) = 0$, i.e., $\lambda^2 - 2\lambda + 1 - a^2 = 0$, so the roots are

$$\lambda = \frac{2 \pm \sqrt{4 - 4 + 4a^2}}{2} = 1 \pm a$$

Thus, both eigenvalues are real, and for any value of a at least one of them is positive, so there are **no** values of a for which all the trajectories flow toward the origin.

In fact, for $-1 < a < 1$, both eigenvalues are positive and the origin is an unstable node. If $a > 1$ or $a < -1$, then there is an eigenvalue of each sign and the origin is a saddle point, and if $a = \pm 1$ then every point on the line $y = -2x$ will be a critical (equilibrium) point. In this case, all the trajectories on this line will neither flow towards or out of the origin.

(b) For what values of a, if any, do all trajectories flow out of the origin (i.e., $\lim_{t\to-\infty} x(t) = (0,0)$ for all trajectories, a.k.a. solutions to they system, x)?

If $-1 < a < 1$ then both eigenvalues are positive, so the origin will be an unstable node .

(c) For what values of a, if any, do some trajectories flow out of the origin and all other trajectories flow towards the origin?

By the above analysis, there are no such values of a.

6. Consider a system

$$x' = 2x(x-2)(y+3)$$
$$y' = (x+2)(y+4)$$

Find all critical points and describe the behavior of the system near these critical points.

For $x' = 0$ we need either $x = 0$, $x = 2$ or $y = -3$. If $x = 0$ or $x = -2$, then the second equation says $y = -4$. If $y = -3$, then the second equation says $x = -2$. Thus there are three critical points: $(0, -4)$, $(2, -4)$ and $(-2, -3)$.

The Jacobian of the system is

$$J = \begin{bmatrix} (4x-4)(y+3) & 2x(x-2) \\ y+4 & x+2 \end{bmatrix}$$

We have $J(0, -4) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ so $(0, -4)$ is an unstable note (all trajectories near $(0, -4)$ flow away from it).

Next, $J(2, -4) = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$, so $(2, -4)$ is a saddle point.

And $J(-2, -3) = \begin{bmatrix} 0 & 16 \\ 1 & 0 \end{bmatrix}$, which has eigenvalues ± 4, so this is also a saddle point.

7. The Texas state legislature decides there are too many medical procedures going on in their state that they don't approve of. They decide to impose some new conditions on which procedures can be done based on the following three variables:

- The time t in weeks that a person has needed the procedure.
- The net wealth w in hundreds of thousands of dollars of the person seeking the procedure.
- The contributions d in thousands of dollars the person has made to members of the legislature in the past year.

The legislature passes a law saying that the vector $v = (t, w, d)$ corresponding to the person requesting the procedure must be at distance at most 10 from the subspace W of $\mathbb{R}^3$ spanned by the vectors $w_1 = (0, 1, 1)$ and $w_2 = (0.1, 1, 0)$. If $v = (10, 8, 10)$, will the procedure be allowed?

Ignoring all the verbiage, the problem is asking whether the (perpendicular) distance from the point v to the subspace W spanned by w_1 and w_2 is greater than or less than 10. This distance is the length of the vector $v - \mathrm{proj}_W\, v$. To calculate the projection, we first need an orthogonal basis for W, so we'll replace w_2 with

$$\widehat{w_2} = w_2 - \frac{\langle w_2 \,,\, w_1 \rangle}{\langle w_1 \,,\, w_1 \rangle} w_1 = (0.1, 1, 0) - \frac{1}{2}(0, 1, 1) = (0.1, 0.5, -0.5)$$

Then

$$\begin{aligned} v - \mathrm{proj}_W\, v &= v - \frac{\langle v \,,\, w_1 \rangle}{\langle w_1 \,,\, w_1 \rangle} w_1 - \frac{\langle v \,,\, \widehat{w_2} \rangle}{\langle \widehat{w_2} \,,\, \widehat{w_2} \rangle} \widehat{w_2} \\ &= (10, 8, 10) - \frac{18}{2}(0, 1, 1) - (0, 0, 0) = (10, -1, 1) \end{aligned}$$

This vector clearly has length bigger than 10 (it's $\sqrt{102}$), so the procedure will **not** be allowed.

8. Suppose $y(t)$ is the displacement of a door to the U.S. Capitol at time t. As a result of the strength of the door and rioters pushing on it, $y(t)$ satisfies the differential equation

$$y''(t) = -y(t) + f(t)$$

where $f(t)$ results from the force applied by the rioters. A quick-thinking Capitol police officer realizes three things:

- $y(0) = 0$ and $y'(0) = 1$.
- The door will break when $y(t) = 1$.
- Reinforcements won't arrive till $t = 1$. For $0 \leqslant t \leqslant 1$ the rioters will maintain $f(t) = 1$. It's not clear what $f(t)$ will be for $t \geqslant 1$.

Fortunately, the officer took Math 2400. From the above information, can she decide whether the door will break before $t = 1$? If so, will it break or not before $t = 1$? Justify your answers.

Again, looking past the blather, the problem is asking whether or not the solution $y(t)$ of the initial-value problem $y'' + y = 1$, $y(0) = 0$, $y'(0) = 1$ will be bigger than 1 for any value of t between 0 and 1. It's not hard to see (by undetermined coefficients) that the general solution of $y'' + y = 1$ is $y = 1 + c_1 \cos t + c_2 \sin t$. For $y(0) = 0$ we need $c_1 = -1$ and for $y'(0) = 1$ we need $c_2 = 1$. So the solution is

$$y(t) = 1 - \cos t + \sin t.$$

We'll have $y(t) = 1$ for $t = \pi/4$, which is less than 1. So it looks like the door will break.

MATH 240 - Final Exam

5/4/2021

General Instructions:

- You have **two hours** to do the quiz. After that, you have 20 minutes to upload your work.
- **In every solution, justification is needed for full points.**
- If it is taking too long to upload your work, let the proctor know in the chat. If you don't, you may not receive points for your work.
- You must remain on Zoom with your camera on until you finish uploading your work.
- You are **not** allowed to consult anything, with the exception of one cheat sheet prepared beforehand.
- Calculators are not allowed.

Problem 1. Let L be a linear transformation from $\mathbb{R}^3$ to $\mathcal{P}_2$, where $\mathcal{P}_2$ is the space of polynomials of degree up to 2, such that:

$$L\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 1 + x, \quad L\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = x^2, \quad L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = 1 + x + 2x^2.$$

(a) Compute $L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ and $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$.
(b) Determine the rank and nullity of L.

Problem 2. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be a linear transformation, where $\mathcal{P}_2$ is the space of polynomials of degree up to 2. Recall that $\mathcal{S} = \{1, x, x^2\}$ is the standard basis of $\mathcal{P}_2$. Let $\mathcal{B} = \{1, x - x^2, 4x^2 - 3x\}$ be another basis. We are given the matrix of T in the basis $\mathcal{B}$:

$$[T]_{\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & -1\end{bmatrix}.$$

Compute $[T]_{\mathcal{S}}$, the matrix of T in the basis $\mathcal{S}$.

Problem 3. Suppose A is a 3×3-matrix with characteristic polynomial given by $\chi_A(\lambda) = -(\lambda - 3)(\lambda - 1)^2$. Suppose the nullity of $(A - I)$ is 1 and the nullity of $(A - I)^2$ is 2.
(a) Find the Jordan canonical form J of A.
(b) Find the general solution of the homogeneous system $x'(t) = Jx(t)$, where J is the Jordan form of A that you found and $x(t)$ is a 3-vector.

Problem 4. Let A be a 2×2-matrix. Suppose there exists a diagonal matrix D and an invertible matrix Q such that $A = QDQ^{-1}$ where:

$$Q = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

We are told that $\mathsf{det}(A) = 0$ and that $A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

(a) Prove that Q is an orthogonal matrix.
(b) Recover the matrix A explicitly.

Problem 5. Consider the nonlinear autonomous system

$$\begin{cases} x' = (x-y)(x-3) \\ y' = xy(y-1) \end{cases}$$

(a) Find all of its critical points.
(b) Find the linearization of this system at the critical point $(3, 1)$, and decide whether $(3, 1)$ is stable or unstable.

Problem 6. Find the general solution to the system $x'(t) = Ax(t)$, where

$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$$

Express your solution in terms of real-valued functions.

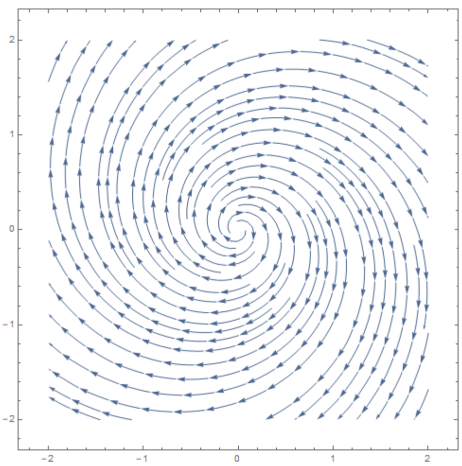
Problem 7. Find the unique solution to the initial value problem:

$$\begin{cases} x^2 y''(x) - 2xy'(x) + 2y(x) = 10x \\ y(1) = 0 \\ y'(1) = 5 \end{cases}$$

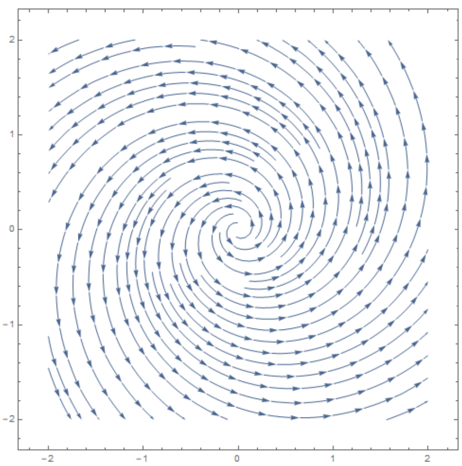
↓ **Last problem on the next page** ↓

Problem 8. Consider the linear system $x'(t) = Ax(t)$, where A is a 2×2 matrix with constant coefficients.

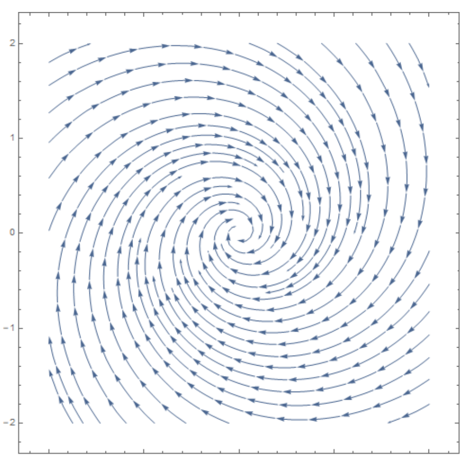
(a) If A has eigenvalues $1 \pm 3i$ and its first column is the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which of the following phase portraits represents the trajectories of the solutions to this system? Justify your answer.



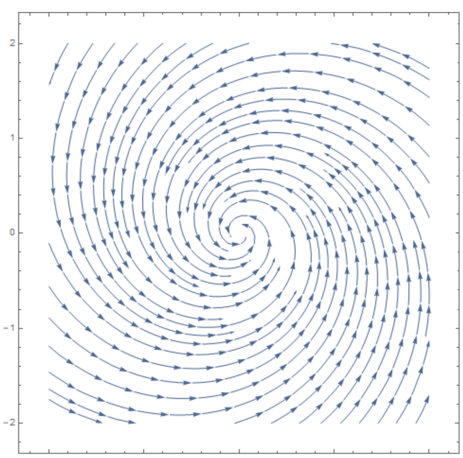
Stream Plot 1



Stream Plot 2



Stream Plot 3



Stream Plot 4

(b) Now consider the case where A has an eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalue $\lambda_1 = -3$, and an eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to eigenvalue $\lambda_2 = 5$. With this information, sketch the phase portrait of the system.

FINAL EXAM SOLUTIONS

№1 [4 pts total = (a) 2 + (b) 2]

(a) $L\begin{bmatrix}0\\1\\0\end{bmatrix} = L\begin{bmatrix}1\\1\\1\end{bmatrix} - L\begin{bmatrix}1\\0\\1\end{bmatrix} = (1+x+2x^2) - (1+x) = \boxed{2x^2}$

$L\begin{bmatrix}1\\0\\0\end{bmatrix} = L\begin{bmatrix}1\\1\\0\end{bmatrix} - L\begin{bmatrix}0\\1\\0\end{bmatrix} = x^2 - 2x^2 = \boxed{-x^2}$

(b) $\operatorname{rk} L = \dim \operatorname{Col} L = \dim \operatorname{Span}(1+x,\ x^2,\ 1+x+2x^2)$

Notice that $1+x+2x^2 = (1+x) + 2\cdot x^2 \in \operatorname{Span}(1+x, x^2)$
and $1+x,\ x^2$ are lin. indep.
$\Longrightarrow$ $1+x,\ x^2$ form a basis for $\operatorname{Col} L$

$\Longrightarrow$ $\boxed{\operatorname{rk} L = \dim \operatorname{Col} L = 2}$

By rank-nullity theorem:

$\boxed{\operatorname{nul} L = \dim \mathbb{R}^3 - \operatorname{rk} L = 3 - 2 = 1}$

№2 $P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{pmatrix}\text{columns are vectors of } \mathcal{B} \\ \text{expressed in terms of } \mathcal{S}\end{pmatrix} = \begin{bmatrix}1&0&0\\0&1&-3\\0&-1&4\end{bmatrix}$

$P_{\mathcal{B}\leftarrow\mathcal{S}} = P_{\mathcal{S}\leftarrow\mathcal{B}}^{-1} = \left[\begin{array}{c|cc}1&0&0\\ \hline 0&1&-3\\0&-1&4\end{array}\right]^{-1} = \left[\begin{array}{c|cc}1&0&0\\ \hline 0&4&3\\0&1&1\end{array}\right]$

the matrix is block-diagonal, so we can invert each block separately

$[T]_{\mathcal{S}} = P_{\mathcal{S}\leftarrow\mathcal{B}}\,[T]_{\mathcal{B}}\,P_{\mathcal{B}\leftarrow\mathcal{S}} = \begin{bmatrix}1&0&0\\0&1&-3\\0&-1&4\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&-1\end{bmatrix}P_{\mathcal{B}\leftarrow\mathcal{S}} =$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 7 \\ 0 & -9 & -8 \end{bmatrix}}$$

№3 [4 pts = (a) 2 + (b) 2]

(a) E-values are roots of char. poly: 3 w/ alg. mult. 1,
1 w/ alg. multiplicity 2

(Geom. multiplicity of 1) = nul(A − I) = 1
⟹ we will have a Jordan block

The only possibility is that the block is 2×2

$$J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $x' = Jx$. General solution is $x = e^{tJ}c$, $c \in \mathbb{R}^3$

$$e^{t\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}$$

№4 [4 pts = (a) 1 + (b) 3]

(a) Q is orthogonal $\iff$ $Q^T Q = I$

So calc. $Q^T Q = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} =$

$$= \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I \quad \checkmark$$

(b) $\begin{bmatrix}2\\4\end{bmatrix} = A\begin{bmatrix}0\\1\end{bmatrix} = \text{col}_2 A \implies A = \begin{bmatrix}? & 2\\ ? & 4\end{bmatrix}$

$A = QDQ^{-1}$ with D diagonal & Q orthog.
$\implies A$ is symm.

Can also check directly: $Q^{-1} = Q^T$ since Q is orthogonal,
$A^T = (QDQ^T)^T = Q^{TT} D^T Q^T = QDQ^T = A \implies A$ symm.

A is symm. $\implies A_{21} = A_{12} = 2 \implies A = \begin{bmatrix}A_{11} & 2\\ 2 & 4\end{bmatrix}$

$0 = \det A = A_{11}\cdot 4 - 4$, so $A_{11} = 1$

$$\boxed{A = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}}$$

№5 [4 pts = (a) 2 + (b) 2]

(a) $\begin{cases}(x-y)(x-3)=0\\ xy(y-1)=0\end{cases}$ $\begin{cases}x=y \text{ or } x=3\\ x=0 \text{ or } y=0 \text{ or } y=1\end{cases}$

Crit. points: $\boxed{(0,0),\ (3,0),\ (1,1),\ (3,1)}$

(b) $F(x,y) = x^2 - 3x - xy + 3y$
$G(x,y) = xy^2 - xy$

$J = \begin{bmatrix}F_x & F_y\\ G_x & G_y\end{bmatrix} = \begin{bmatrix}2x-3-y & -x+3\\ y^2-y & 2xy-x\end{bmatrix}$, $J(3,1) = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

Two distinct positive e-values (2 and 3) $\implies$
improper source node $\implies$ unstable

OR: $\text{tr} > 0$ & $\det > 0 \implies$ unstable

№6

$$\chi_A(\lambda) = \begin{vmatrix} 3-\lambda & -5 \\ 5 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 + 25 = \lambda^2 - 6\lambda + 34$$

Quadratic formula: $D = 36 - 4\cdot 34 = -4\cdot 25$, $\lambda_\pm = 3 \pm 5i$

Enough to work with 1 e-vector, so pick one e-value:

$\lambda_+ = 3+5i$: $\quad \operatorname{Ker}(A - \lambda_+ I) = \operatorname{Ker}\begin{bmatrix} -5i & -5 \\ 5 & -5i \end{bmatrix} \ni \begin{bmatrix} i \\ 1 \end{bmatrix} =: v$

The general solution is $c_1 \cdot \operatorname{Re}(e^{\lambda_+ t} v) + c_2 \cdot \operatorname{Im}(e^{\lambda_+ t} v)$, so calculate

$$e^{\lambda_+ t} v = e^{(3+5i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{3t} \cdot (\cos 5t + i \sin 5t) \begin{bmatrix} i \\ 1 \end{bmatrix} =$$

$$= e^{3t} \begin{bmatrix} -\sin 5t + i\cos 5t \\ \cos 5t + i \sin 5t \end{bmatrix}$$

$$\boxed{x_g(t) = c_1 e^{3t} \begin{bmatrix} -\sin 5t \\ \cos 5t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \cos 5t \\ \sin 5t \end{bmatrix}} \text{ OR } \boxed{= e^{3t} \begin{bmatrix} -\sin 5t & \cos 5t \\ \cos 5t & \sin 5t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

№7 $\quad x^2 y'' - 2xy' + 2y = 10x$

This is a Cauchy-Euler equation, so a solution to the homogeneous version is of the form $y(x) = x^p$.

Equation for p:

$$p(p-1) - 2p + 2 = 0$$
$$p^2 - 3p + 2 = 0$$
$$p = 1 \text{ or } p = 2$$

So general sol. to homog. equation is $y_h(x) = c_1 x + c_2 x^2$

Note that RHS $= 10x$ is a solution to the homog. equation, so a particular solution to the inhomog. equation should be:

$y_p(x) = A x \ln x \quad$ for some $A \in \mathbb{R}$.

$y_p'(x) = A \ln x + A$

$y_p''(x) = A \cdot \frac{1}{x}$

Plug this into the inhomog. eq'n to find A:

$Ax - 2x \cdot (A \cancel{\ln x} + A) + \cancel{2Ax\ln x} = 10x$

$-Ax = 10x \Rightarrow A = -10$

So the general sol. is $y_g(x) = -10x\ln x + c_1 x + c_2 x^2$

Substitute initial values:

$$\begin{cases} 0 = y(1) = -10 \cdot 0 + c_1 + c_2 \\ 5 = y'(1) = -10 \cdot 0 - 10 + c_1 + 2c_2 \end{cases} \qquad \begin{cases} c_1 + c_2 = 0 \\ c_1 + 2c_2 = 15 \end{cases}$$

$\Longrightarrow c_2 = 15, \quad c_1 = -15$

Answer: $\boxed{y(x) = -10x\ln x - 15x + 15x^2}$

№8 [4 total = (a)2 + (b)2]

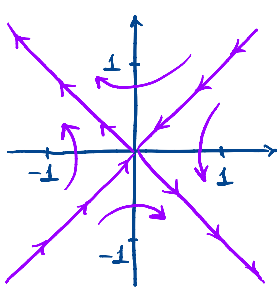
(a) Complex e-values $\Rightarrow$ spiral

$\mathrm{Re}(1 \pm 3i) = 1 > 0 \Longrightarrow$ source

$\mathrm{col}_1(A) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ above the x-axis $\Rightarrow$ counterclockwise

Answer: Stream Plot 2.

(b) Two e-values of opposite signs $\Rightarrow$ saddle.



MATH 240 – Final Exam – Fall 2020

You have 100 minutes to complete this exam, which consists of eight questions.

Besides this document, a writing implement, paper and your brain, you may use only a single standard-size sheet of paper with notes written on both sides. No other resources or technology is allowed while you are working this exam.

You may choose to print out the exam and show your work and answers on it. Otherwise, please use a separate sheet of paper for each problem.

When you are finished, please scan your work into a single pdf file and upload it back to the Final Exam page in the Week 14 module on the Canvas site.

By accessing this exam, you are agreeing to abide Penn's Code of Academic Integrity. In particular you agree not to consult any resources other than those enumerated above, and to submit work that is entirely your own.

You should have plenty of time to complete this exam — take your time and do your best.

Good luck!

1. Let L be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix} \quad L\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix} \quad L\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix} \quad L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)$

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$ and $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$

(c) What is the rank of L (which is the same as the rank of M_L)? How do you know?

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

2. Let $M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$

(a) What are the rank and nullity of M?

(b) What are the eigenvalues of M?

(c) Is M diagonalizable? Explain how you know.

3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

4. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, and let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{x}$, with $\mathbf{v} \in \mathcal{S}$ and $\mathbf{w}$ perpendicular to $\mathcal{S}$ (with respect to the standard inner product on $\mathbb{R}^4$).

5. An undamped mass/spring system satisfies the differential equation $y'' + 16y = 0$.

(a) If the mass is released from rest, 0.2 meters from its equilibrium point, how long until it returns to its starting point (for the first time after being released)?

(b) If the mass is released from rest, 0.4 meters from its equilibrium point, now how long until it returns to its starting point (for the first time)?

(c) Now suppose the mass is doubled and it is released from rest, 0.2 meters from its equilibrium point. Now how long until it returns to its starting point?

6. (a) Find the general solution of the equation $y'' - 2y' + y = 0$.

(b) Find the Wronskian of the two independent solutions of $y'' - 2y' + y = 0$ that you found in part (a).

(c) Find the general solution of the equation $y'' - 2y' + y = \dfrac{e^t}{t+4}$.

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

(b) Sketch the phase portrait for the system $\mathbf{x}' = A\mathbf{x}$, drawing several representative trajectories and indicating their direction.

8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = (2 - x)(4 - y)$$

$$\frac{dy}{dt} = x(1 + y)$$

MATH 240 – Final Exam – Fall 2020

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Good luck!

MATH 240 –Final exam – Fall 2020

1. Let L be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}\quad L\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}\quad L\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}\quad L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)$

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$ and $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$

(c) What is the rank of L? How do you know?

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

2. Let $M = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 3 & 0 & -3 & 0 & 3 & 0 \\ 3 & 0 & -3 & 0 & 3 & 0 \end{bmatrix}$

(a) What are the rank and nullity of M?

(b) What are the eigenvalues of M?

(c) Is M diagonalizable? Explain how you know.

3. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

(d) Using your answer to parts (a), (b) and (c) , write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

4. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, and let $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{x}$, with $\mathbf{v} \in \mathcal{S}$ and $\mathbf{w}$ perpendicular to $\mathcal{S}$ (with respect to the standard inner product on $\mathbb{R}^4$).

5. An undamped mass/spring system satisfies the differential equation $y'' + 36y = 0$.

(a) If the mass is released from rest, 0.5 meters from its equilibrium point, how long until it returns to its starting point (for the first time after being released)?

(b) If the mass is released from rest, 0.4 meters from its equilibrium point, now how long until it returns to its starting point (for the first time)?

(c) Now suppose the mass is doubled and it is released from rest, 0.5 meters from its equilibrium point. Now how long until it returns to its starting point?

6. (a) Find the general solution of the equation $y'' + 2y' + y = 0$.

(b) Find the Wronskian of the two independent solutions of $y'' + 2y' + y = 0$ that you found in part (a).

(c) Find the general solution of the equation $y'' + 2y' + y = \dfrac{e^{-t}}{t-4}$.

7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

(b) Sketch the phase portrait for the system $\mathbf{x}' = A\mathbf{x}$, drawing several representative trajectories and indicating their direction.

8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = y(1 + x)$$

$$\frac{dy}{dt} = (4 - x)(2 - y)$$

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MATH 240 –Final exam – Fall 2020

1. Let L be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}\qquad L\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}\qquad L\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}\qquad L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)$

Since $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Since $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Since $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}-\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$.

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$ and $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$

The columns of M_L are the images of the (standard) basis vectors and we have three of them. We still need to observe that since $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Therefore

$$M_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

(c) What is the rank of L (which is the same as the rank of M_L)? How do you know?

The image of L contains a basis of $\mathbb{R}^2$ (the middle two given facts are the standard basis of $\mathbb{R}^2$) so the rank of L is 2.

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

By the rank-nullity theorem the dimension of the kernel is $4 - 2 = 2$.

2. Let $M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$

(a) What are the rank and nullity of M?

Since every row of M is a multiple of the first (nonzero) row, the rank of M is 1. Therefore the nullity is $6 - 1 = 5$.

(b) What are the eigenvalues of M?

$\lambda = 0$ is an eigenvalue of M with multiplicity 5. The other eigenvalue of M is tr$(M) = 12$.

(c) Is M diagonalizable? Explain how you know.

M is diagonalizable because $\mathbb{R}^6$ has a basis of eigenvectors of M – the five vectors that span the nullspace of M plus an eigenvector for $\lambda = 12$.

3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

The eigenvalues of A are $\lambda = 2$ and $\lambda = 1$ (with multiplicity 2).
For $\lambda = 2$, we have $A - 2\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.
For $\lambda = 1$, we have $A - \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, which has rank 2, so there is only one linearly independent eigenvector, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is $J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

We have the eigenvectors for $\lambda = 2$ and $\lambda = 1$, and we need a generalized eigenvector for $\lambda = 1$.
Since $(A - \mathbf{I})^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then
$\mathbf{v}_2 = (A - \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and so $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

The solution of $\mathbf{y}' = J\mathbf{y}$ is $\mathbf{y} = c_1e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3e^{t}\begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$. Or, you could say that
$e^{tJ} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

The solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = P\mathbf{y} = c_1e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + c_3e^{t}\begin{bmatrix} 0 \\ 1 \\ -t \end{bmatrix}$.

4. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, and let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{x}$, with $\mathbf{v} \in \mathcal{S}$ and $\mathbf{w}$ perpendicular to $\mathcal{S}$ (with respect to the standard inner product on $\mathbb{R}^4$).

It's not hard to see that an orthonormal basis for $\mathcal{S}$ is $\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Then

$$\mathbf{v} = \mathrm{proj}_{\mathcal{S}}\, \mathbf{x} = \langle \mathbf{x}\,,\, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}\,,\, \mathbf{e}_2 \rangle \mathbf{e}_2 = \frac{1}{3} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

And then

$$\mathbf{w} = \mathbf{x} - \mathbf{v} = \frac{1}{3} \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

5. An undamped mass/spring system satisfies the differential equation $y'' + 16y = 0$.

(a) If the mass is released from rest, 0.2 meters from its equilibrium point, how long until it returns to its starting point (for the first time after being released)?

The motion will satisfy $y(t) = 0.2\cos(4t)$. The mass will return to 0.2 meters when $4t = 2\pi$, or $\pi/2$ seconds later.

(b) If the mass is released from rest, 0.4 meters from its equilibrium point, now how long until it returns to its starting point (for the first time)?

Now the motion will satisfy $y(t) = 0.4\cos(4t)$, so the answer is still $\pi/2$ seconds later.

(c) Now suppose the mass is doubled and it is released from rest, 0.2 meters from its equilibrium point. Now how long until it returns to its starting point?

This changes the differential equation to $2y'' + 16y = 0$, or $y'' + 8y = 0$. so the solution is $y = 0.2\cos(\sqrt{8}\,t)$, so the mass returns to the starting point when $\sqrt{8}\,t = 2\pi$, or $\pi/\sqrt{2}$ seconds later.

6. (a) Find the general solution of the equation $y'' - 2y' + y = 0$.

$r^2 - 2r + 1 = (r-1)^2$ so the general solution is $y = c_1 e^t + c_2 t e^t$.

(b) Find the Wronskian of the two independent solutions of $y'' - 2y' + y = 0$ that you found in part (a).

With $y_1 = e^t$ and $y_2 = te^t$, $W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$.

(c) Find the general solution of the equation $y'' - 2y' + y = \dfrac{e^t}{t+4}$.

Variation of parameters.

$$\begin{aligned} y &= y_1 \int \frac{-y_2 f}{W}\, dt + y_2 \int \frac{y_1 f}{W}\, dt \\ &= e^t \int \frac{-te^t}{e^{2t}} \frac{e^t}{t+4}\, dt + te^t \int \frac{e^t}{e^{2t}} \frac{e^t}{t+4}\, dt \\ &= e^t \int \frac{-t}{t+4}\, dt + te^t \int \frac{1}{t+4}\, dt \\ &= e^t \int \frac{4}{t+4} - 1\, dt + te^t(\ln(t+4) + c_2) \\ &= e^t(4\ln(t+4) - t + c_1) + te^t \ln(t+4) + c_2 te^t \\ &= 4e^t \ln(t+4) - te^t + te^t \ln(t+4) + c_1 e^t + c_2 te^t \end{aligned}$$

(You could absorb the $-te^t$ into the c_2 term and simplify to $y = (t+4)e^t \ln(t+4) + c_1 e^t + c_2 te^t$)

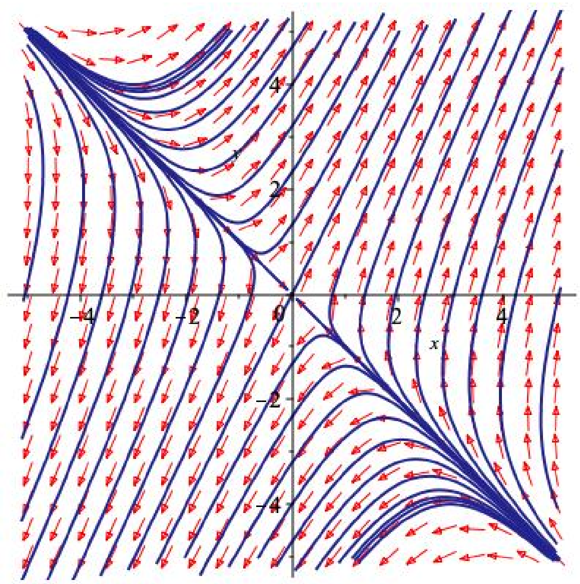
7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since $\det(A) = -5$, the origin is a saddle point. Also, the sum of both columns is 5, so that is one of the eigenvalues of A, and the other is -1, since the trace of A is 4.

(b) Sketch the phase portrait for the system $\mathbf{x}' = A\mathbf{x}$, drawing several representative trajectories and indicating their direction.

The eigenvector for $\lambda = 5$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ – trajectories go out along this vector (and its negative). The eigenvector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and trajectories come in along this vector.



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = (2-x)(4-y)$$

$$\frac{dy}{dt} = x(1+y)$$

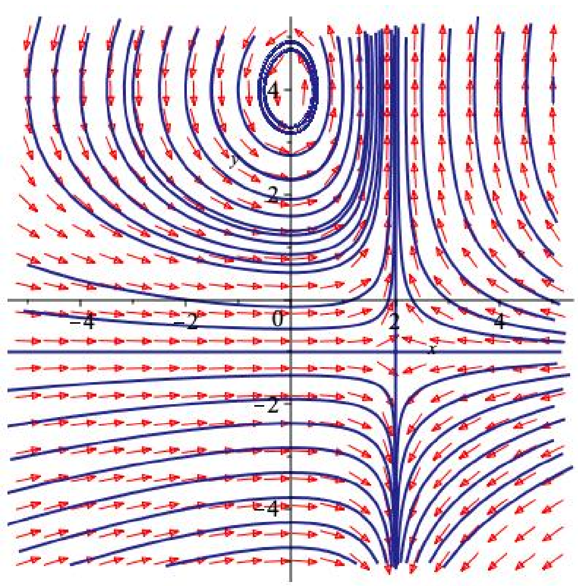
For the critical points, we need both right-hand sides to be zero. The first one is zero if $x = 2$ or $y = 4$ and the second if $x = 0$ or $y = -1$. Therefore there are two critical points $(2, -1)$ and $(0, 4)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} y-4 & x-2 \\ y+1 & x \end{bmatrix}$.

At $(2, -1)$ we have $J(2, -1) = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$, which has negative determinant so $(2, -1)$ is a saddle point.

At $(0, 4)$, $J(0, 4) = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}$, which is a center.

For the record:



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1. Let L be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}\qquad L\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}\qquad L\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}\qquad L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)$

Since $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$.

Since $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$.

Since $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}-\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}0\\-1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$.

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$ and $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$

The columns of M_L are the images of the (standard) basis vectors and we have three of them. We still need to observe that since $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Therefore

$$M_L = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 2 & -1 & -2 & 1 \end{bmatrix}$$

(c) What is the rank of L? How do you know?

The image of L contains a basis of $\mathbb{R}^2$ (the middle two given facts are the standard basis of $\mathbb{R}^2$) so the rank of L is 2.

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

By the rank-nullity theorem the dimension of the kernel is $4 - 2 = 2$.

2. Let $M = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 3 & 0 & -3 & 0 & 3 & 0 \\ 3 & 0 & -3 & 0 & 3 & 0 \end{bmatrix}$

(a) What are the rank and nullity of M?

Since every row of M is a multiple of the first (nonzero) row, the rank of M is 1. Therefore the nullity is $6 - 1 = 5$.

(b) What are the eigenvalues of M?

$\lambda = 0$ is an eigenvalue of M with multiplicity 5. The other eigenvalue of M is tr$(M) = 2$.

(c) Is M diagonalizable? Explain how you know.

M is diagonalizable because $\mathbb{R}^6$ has a basis of eigenvectors of M – the five vectors that span the nullspace of M plus an eigenvector for $\lambda = 2$.

3. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

The eigenvalues of A are $\lambda = 3$ and $\lambda = 1$ (with multiplicity 2).

For $\lambda = 3$, we have $A - 3\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

For $\lambda = 1$, we have $A - \mathbf{I} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, which has rank 2, so there is only one linearly independent eigenvector, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

We have the eigenvectors for $\lambda = 3$ and $\lambda = 1$, and we need a generalized eigenvector for $\lambda = 1$.

Since $(A - \mathbf{I})^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}_2 = (A - \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and so $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

The solution of $\mathbf{y}' = J\mathbf{y}$ is $\mathbf{y} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$. Or, you could say that $e^{tJ} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$

(d) Using your answer to parts (a), (b) and (c) , write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

The solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = P\mathbf{y} = c_1 e^{3t} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix}$.

4. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, and let $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{x}$, with $\mathbf{v} \in \mathcal{S}$ and $\mathbf{w}$ perpendicular to $\mathcal{S}$ (with respect to the standard inner product on $\mathbb{R}^4$).

It's not hard to see that an orthonormal basis for $\mathcal{S}$ is $\left\{ \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{e}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then

$$\mathbf{v} = \operatorname{proj}_{\mathcal{S}} \mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

And then

$$\mathbf{w} = \mathbf{x} - \mathbf{v} = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

5. An undamped mass/spring system satisfies the differential equation $y'' + 36y = 0$.

(a) If the mass is released from rest, 0.5 meters from its equilibrium point, how long until it returns to its starting point (for the first time after being released)?

The motion will satisfy $y(t) = 0.5\cos(6t)$. The mass will return to 0.5 meters when $6t = 2\pi$, or $\pi/3$ seconds later.

(b) If the mass is released from rest, 0.4 meters from its equilibrium point, now how long until it returns to its starting point (for the first time)?

Now the motion will satisfy $y(t) = 0.4\cos(6t)$, so the answer is still $\pi/3$ seconds later.

(c) Now suppose the mass is doubled and it is released from rest, 0.5 meters from its equilibrium point. Now how long until it returns to its starting point?

This changes the differential equation to $2y'' + 36y = 0$, or $y'' + 18y = 0$. so the solution is $y = 0.5\cos(\sqrt{18}\,t)$, so the mass returns to the starting point when $\sqrt{18}\,t = 2\pi$, or $\dfrac{\sqrt{2}\,\pi}{3}$ seconds later.

6. (a) Find the general solution of the equation $y'' + 2y' + y = 0$.

$r^2 - 2r + 1 = (r-1)^2$ so the general solution is $y = c_1 e^{-t} + c_2 t e^{-t}$.

(b) Find the Wronskian of the two independent solutions of $y'' + 2y' + y = 0$ that you found in part (a).

With $y_1 = e^{-t}$ and $y_2 = te^{-t}$, $W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t}$.

(c) Find the general solution of the equation $y'' + 2y' + y = \dfrac{e^{-t}}{t-4}$.

Variation of parameters.

$$\begin{aligned}
y &= y_1 \int \frac{-y_2 f}{W}\,dt + y_2 \int \frac{y_1 f}{W}\,dt \\
&= e^{-t} \int \frac{-te^{-t}}{e^{-2t}} \frac{e^{-t}}{t-4}\,dt + te^{-t} \int \frac{e^{-t}}{e^{-2t}} \frac{e^{-t}}{t-4}\,dt \\
&= e^{-t} \int \frac{-t}{t-4}\,dt + te^{-t} \int \frac{1}{t-4}\,dt \\
&= e^{-t} \int \frac{-4}{t-4} - 1\,dt + te^{-t}(\ln(t-4) + c_2) \\
&= e^{-t}(-4\ln(t-4) - t + c_1) + te^{-t}\ln(t-4) + c_2 te^{-t} \\
&= -4e^{-t}\ln(t-4) - te^{-t} + te^{-t}\ln(t-4) + c_1 e^{-t} + c_2 te^{-t}
\end{aligned}$$

(You could absorb the $-te^t$ into the c_2 term and simplify to $y = (t-4)e^{-t}\ln(t-4) + c_1 e^{-t} + c_2 te^{-t}$)

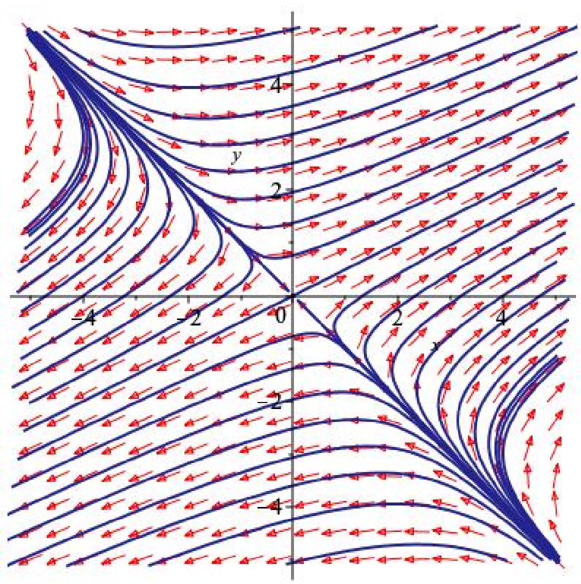
7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since det$(A) = -5$, the origin is a saddle point. Also, the sum of both columns is 5, so that is one of the eigenvalues of A, and the other is -1, since the trace of A is 4.

(b) Sketch the phase portrait for the system $\mathbf{x}' = A\mathbf{x}$, drawing several representative trajectories and indicating their direction.

The eigenvector for $\lambda = 5$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ – trajectories go out along this vector (and its negative). The eigenvector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and trajectories come in along this vector.



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = y(1+x)$$

$$\frac{dy}{dt} = (4-x)(2-y)$$

For the critical points, we need both right-hand sides to be zero. The first one is zero if $x = -1$ or $y = 0$ and the second if $x = 4$ or $y = -2$. Therefore there are two critical points $(-1, 2)$ and $(4, 0)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} y & x+1 \\ y-2 & x-4 \end{bmatrix}$.

At $(-1, 2)$ we have $J(-1, 2) = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$, which has negative determinant so $(-1, 2)$ is a saddle point.

At $(4, 0)$, $J(4, 0) = \begin{bmatrix} 0 & 5 \\ -2 & 0 \end{bmatrix}$, which is a center.

For the record: