8 Interactions between linear transformations and inner products; symmetric matrices; Singular Value Decomposition

8.1 Inner products and linear transformations, self-adjoint transformations and symmetric matrices

Now that we have introduced the ideas of linear transformations and inner products, we can discuss the way that both concepts can interact with each other.

- When a linear transformation preserves the inner product between two vectors, it is called an **orthogonal transformation**
  - Let $T : V \to W$ and $q, p \in V$ and $T(q), T(p) \in W$, we know $< q, p >=< T(q), T(p) >$
  - For matrices that obey this property, we know about the matrix, A
    - $Det(A) = \pm 1$
    - $(Ap)^T(Aq) = p^T(A^TA)q$
    - $A^TA = I$
    - **Ais an orthogonal matrix** – each of its columns are mutually orthogonal
- A similar type of linear transformation is the **self-adjoint transformation**
  - Let $T : V \to W$ and $q, p \in V$ and $T(q), T(p) \in W$, we know $< T(q), p >=< q, T(p) >$
  - $< Aq, p >=< q, Ap >\to q^TAp = q^TAp \to A^T = A$
  - **A is a symmetric matrix**
  - $NullSp(A)$ and $ColSp(A)$ are *orthogonal spaces*, meaning every single vector in the nullspace is orthogonal to every single vector in the column space
  - If v is an eigenvector of A, and w is any vector orthogonal to v, then Aw is also orthogonal to v
- Self-adjoint transformations and symmetric matrices have specific properties, which is summarized as the **Spectral Theorem**:
  - A *always* has real eigenevalues
  - A is *always* diagonalizable
  - Eigenvectors with different eigenvalues are orthogonal
    - However, eigenvectors with the eigenvalues are not necessarily orthogonal, so when diagonalizing, we must *make them orthogonal* (see this in examples)
  - $D = P^{-1}AP = P^TAP$ (when diagonalizing, P is orthogonal)
  - The **spectral decomposition** of symmetric matrices is:
    $A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + ... + \lambda_n v_n v_n^T$

    where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A and $v_1, v_2, ..., v_n$ are their respective eigenvectors
    - We also know $Tr(A^TA) = \lambda_1^2 + \lambda_2^2 + ... + \lambda_n^2$

The main questions associated with this section include diagonalizing orthogonal and symmetric matrices, and recalling the Spectral Theorem for applications of symmetric matrices. Let's do a few examples

Orthogonal diagonalization 2×2 example

$$\text{Let } M = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix}.$$

Find a diagonal matrix D and an **orthogonal** matrix P such that $M = PDP^T$ and show its spectral decomposition.

<u>Step 1: Solve for eigenvalues</u>

Since this matrix is symmetric (observe $M = M^T$), we know that we have **real**, not complex, eigenvalues per the Spectral Theorem. Further, we can use some tricks to solve for the eigenvalues. Most notably, since all of the rows and columns sum to 36, we know that must be $\lambda_1 = 36$, and since $tr(M) = 20 + 20 = 40 = \lambda_1 + \lambda_2 = 36 + \lambda_2 \to \lambda_2 = 4$.

We could also solve explicitly:

$$|M - \lambda I| = (20 - \lambda)(20 - \lambda) - 256 = (\lambda - 36)(\lambda - 4) \to \lambda = 36, 4$$

<u>Step 2: Solve for eigenvectors</u>

We know that this matrix is diagonalizable because it is symmetric (Spectral Theorem), so we know we have "enough" eigenvectors.

$\underline{\lambda = 36}$

$$M - 36I = \begin{bmatrix} -16 & 16 \\ 16 & -16 \end{bmatrix}$$

$$\xrightarrow{RREF}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$Rank(M - 36I) = 1,\ Nullity(M - 36I) = 1$$

$$NullSp(M - 36I) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 4}$

$$M - 4I = \begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}$$

$$\xrightarrow{RREF}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$Rank(M - 4I) = 1,\ Nullity(M - 4I) = 1$$

$$NullSp(M - 4I) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

<u>Step 3: Ensure eigenvectors are orthogonal</u>

Here, we know that these two eigenvectors must be orthogonal, per the Spectral Theorem, because they are for two different eigenvalues. This step becomes more important when we have multiple eigenvectors for a single eigenvalue.

We can, however, double check:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1(1) + 1(-1) = 0 \to \text{orthogonal}$$

<u>Step 4: Form D</u>

Forming D is very simple since we are dealing with a diagonalizable system, so we can just place the eigenvalues along the diagonal:

$$D = \begin{bmatrix} 36 & 0 \\ 0 & 4 \end{bmatrix}$$

Step 5: Form P

However, for P, we must take our corresponding eigenvectors from Steps 2/3 and **normalize** them.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \to P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$M = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix} = PDP^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 36 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T$$

Further, we can show M's spectral decomposition:

$$M = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T =$$

$$36 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} =$$

$$36 \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + 4 \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix} = M$$

Let's do one more example.

Orthogonal diagonalization 3×3 example

$$\text{Let } A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Given $\chi(\lambda) = -\lambda(\lambda - 3)^2$, how can we tell that A is diagonalizable just by looking at it? Then, find D and P such that $A = PDP^T$

First, let's answer the first question: How can we tell that A is diagonalizable just by looking at it? **Since A is symmetric, by the Spectral Theorem, A is diagonalizable**

Now, let's solve for P and D

Step 1: Solve for eigenvalues

Since we were provided the characteristic polynomial, we can skip ahead to:

$$\lambda = 0, 3 \text{ (algebraic multiplicity} = 2)$$

(continues on next page)

Step 2: Solve for eigenvectors

A is diagonalizable, so since we have a 3×3, we will have 3 eigenvectors.

$\underline{\lambda = 0}$

$$A - 0I = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{RREF}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Rank(A - 0I) = 2,\ Nullity(A - 0I) = 1$$

$$NullSp(A - 0I) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 3}$

$$A - 3I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{RREF}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Rank(A - 3I) = 1,\ Nullity(A - 3I) = 2$$

$$NullSp(A - 3I) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 3: Ensure eigenvectors are orthogonal

Obviously, by the Spectral Theorem, the eigenvector of $\lambda = 0$ will be orthogonal to both eigenvectors of $\lambda = 3$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1(-1) + 1(1) + 1(0) = 0 \rightarrow \text{orthogonal}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1(-1) + 1(0) + 1(1) = 0 \rightarrow \text{orthogonal}$$

However, for the two eigenvectors of $\lambda = 3$, we don't see this:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1(-1) + 1(0) + 0(1) = 1 \not\rightarrow \text{orthogonal}$$

To ensure that these two vectors are orthogonal, we will use the projection of one vector onto another, and subtract away the projection, just like the Gram Schmidt process.

$$\text{Let } v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We choose to keep v_2 and manipulate v_3. This choice is arbitrary, you could do it the opposite and still get a correct answer.

$$w_3 = v_3 - \frac{<v_2, v_3>}{<v_2, v_2>} v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = w_3$$

Now, we've generated a "new" eigenvector for $\lambda = 3$ that is orthogonal to v_2:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = -1(\tfrac{1}{2}) + 1(\tfrac{1}{2}) + 0(-1) = 0 \to \text{orthogonal}$$

We can even confirm that w_3 is still an eigenvector for $\lambda = 3$:

$$Av = \lambda v = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -3 \end{bmatrix} = 3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Step 4: Form D

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Step 5: Form P

Recall all columns of P must be unit vectors

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \end{bmatrix} \to P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$A = PDP^T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}^T$$

Symmetric matrices and their factorization has a lot of value in linear algebra, especially in finding maximums of certain constraints, and in SVD in 8.2