6 Eigenvalues and eigenvectors; diagonalization

6.1 Linear transformations from V to W versus linear transformation V to V

Now that we have talked about linear transformations in general in the past week, let's talk more about the types of linear transformations you will encounter in this class.
In general, you will be transforming either from one vector space to a different vector space, or you will be transforming items in one vector space to other items within that vector space. When we refer to these, traditionally the first vector space is called "V", and the second, distinct space is called, "W". If you are transforming from one space to another, it would be $V \to W$, but if you are transforming to the same space, it would be $V \to V$.
As we have talked about previously, when we are transforming from one space to another, whether that space is the same or different, we must choose an input and output basis, i.e. a basis to represent the space that you are transforming, and a basis for the results of those transformations. Further, we know that we can represent a linear transformation using a matrix, as seen in 5.2.

Linear transformations from V to W
When transforming from one vector space, V, to another, W, the size of the matrices is important to consider.
Let $dim(V) = n$, which is the dimension of the input vector space, V.
Let $dim(W) = m$, which is the dimension of the output vector space, W.
Let $rank(T) = r$, which is the dimension of the linear transformation. This is the same as the rank of the matrix representing the linear transformation, commonly M_T.
From this, we know that $0 \leq r \leq min(m, n)$. The rank of the transformation must be less than or equal to the smaller dimension of the space.
From rank nullity theorem, we also know that $Nullity(T) = n - r$.
Let the basis matrix for the input vector space be denoted P, and the basis matrix for the output vector space be denoted Q. We also have our linear transformation matrix, M_T. To get the elements in Q, we can simply multiply the elements in P on the left by the matrix M_T:

$$Q = M_T P$$

In all, given a linear transformation matrix, we want to be able to solve for the input and output bases, and given the input and output bases, we want to be able to solve for the linear transformation matrix.
Solve for the input and output bases, P and Q, respectively, and N such that $N = Q^{-1}MP$ example

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$$

Looking at this matrix, which is a 4×3 matrix, we recognize that our linear transformation is $T : \mathbb{R}^3 \to \mathbb{R}^4$, so our input basis, P, should be a 4×4 matrix, and our output basis, Q, should be a 3×3 matrix.

Step 1: Find $RREF(M)$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Rank(M) = 2 \text{ and } Nullity(M) = 1$$

Step 2: Find $NullSp(M)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to NullSp(M) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Step 3: Find basis for $\mathbb{R}^3$ based on M

We place our $NullSp(M)$ basis in our P on the **rightmost** columns, and will fill the remaining two columns with 2 linearly independent 3×1 vectors to generate a basis for $\mathbb{R}^3$

$$P = \begin{bmatrix} & & 1 \\ & & 1 \\ & & -1 \end{bmatrix} \xrightarrow{fill} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Reiterating that these two extra vectors help generate a basis for $\mathbb{R}^3$, and it can be *any two more linearly independent vectors* from $NullSp(M)$. Obviously, if there were more vectors $\in NullSp(M)$, we would add less vectors. Further, it is obvious $NullSp(M)$ will already contain linearly independent vectors since it itself is a basis.

Step 4: Find basis for $\mathbb{R}^4$ based on M

We take our "generated" vectors from P and transform them according to M.

$$q_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \text{ and } q_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}$$

We do not use our $NullSp(M)$ vector here, because it $\in NullSp(M)$, and would only generate the zero vector by definition of nullspaces. We want to form a basis for $\mathbb{R}^4$, so that would not be helpful. These vectors, q_1, q_2, form the **leftmost** vectors in Q, and the remaining columns can be filled with any 2 linearly independent 4×1 vectors to generate a basis for $\mathbb{R}^4$.

$$Q = \begin{bmatrix} 1 & 2 & & \\ 2 & 4 & & \\ 0 & 1 & & \\ 1 & 3 & & \end{bmatrix} \xrightarrow{fill} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}$$

Step 5: Find N, where $N = Q^{-1}MP$

Now that we've solved for P and Q and have known M, we can solve for N. We can manually compute N:

$$N = Q^{-1}MP = \begin{bmatrix}1 & 2 & 1 & 0\\2 & 4 & 0 & 1\\0 & 1 & 0 & 0\\1 & 3 & 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\0 & 1 & 1\\1 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

However, this computation is unnecessary, because my virtue of this method, N will *always* be a matrix the same size and rank as M, with 1's along the diagonal, like a pseudo-identity matrix. This is because we solved for our input and output bases, P and Q, to make this transformation, M, as simple as possible.
This $T : \mathbb{R}^4 \to \mathbb{R}^3$ is both M and N, where
M is the matrix representation of the transformation with respect to the *standard bases of* $\mathbb{R}^4$ *and* $\mathbb{R}^3$, while
N is the matrix representation of the transformation with respect to the *bases of* $\mathbb{R}^4$ *and* $\mathbb{R}^3$ *in* P *and* Q*, respectively.*

Solve for the linear transformation matrix given the input and output bases example

Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ that satisfies:

$$T\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Find the matrix of T with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.

This question should be a lot easier to approach after our previous problem. Here, we are given the inputs and outputs after being transformed by some transformation T, which can be represented by a matrix M_T. Let our inputs be in P and our outputs in Q, just like the previous question:

$$M_T P = Q = M_T \begin{bmatrix}2 & 1 & 1\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\-1 & 2 & 1\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} \to M_T = \begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix}$$

We just now need to solve for this 4×3 matrix,M_T, and we can do this by $M_T P = Q \to M_T P P^{-1} = QP^{-1} = M_T$. Skipping the direct inverse calculation:

$$M_T = QP^{-1} = \begin{bmatrix}1 & 0 & 1\\-1 & 2 & 1\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0 & 1\\-1 & 2 & 1\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\0 & 0 & 1\\-1 & 2 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\-2 & 3 & 4\\-1 & 2 & 2\\0 & 1 & 0\end{bmatrix}$$

$$M_T = \begin{bmatrix}0 & 1 & 0\\-2 & 3 & 4\\-1 & 2 & 2\\0 & 1 & 0\end{bmatrix}$$

We are done!

Linear transformations from V to V
Using the same logic as for $V \to W$:
Let $dim(V) = n$, which is the dimension of the input vector space, V.
Let $dim(W) = n$, which is the dimension of the output vector space, V.
Let $rank(T) = r$, which is the dimension of the linear transformation. This is the same as the rank of the matrix representing the linear transformation, commonly M_T.
From this, we know that $0 \leq r \leq n$. The rank of the transformation must be less than or equal to n.
From rank nullity theorem, we also know that $Nullity(T) = n - r$.

The concept of a linear transformation from one space to the same space introduces us to the idea of **diagonalization**, which is explored fully in 6.2. For now, the introduction of the topic is essentially:

Let V be our $n-$dimensional vector space. We can have infinitely many bases for V, but let B_1 and B_2 be two bases for V. Suppose M_T is the linear transformation matrix such that $T : V \to V$ with respect to the basis B_1. Further, let P be an $n \times n$ change of basis matrix such that $P_{B_2 \to B_1}$ (meaning the columns of P are the basis vectors of B_2 expressed in terms of B_1). This means we can make the matrix N_T, the same representation of $T : V \to V$, with respect to B_2 instead.

$$N_T = P^{-1} M_T P$$

This will make more sense in 6.2 with the introduction of diagonalization, and eigenvales/eigenvectors.