12 More complicated systems of differential equations

12.1 Non-diagonalizable coefficient matrices

Find the fundamental solutions matrix for the following system.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A-\lambda I = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2$$

$\lambda = 2$, alg. mult. $= 2$

$$A-2I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{rank}=1,\ \text{nullity}=1 \quad \ker(A-2I) = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$(A-2I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{rank}=0,\ \text{nullity}=2=\text{multiplicity} \quad \ker((A-2I)^2) = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$v_2 \in \ker((A-2I)^2)$ & $v_2 \notin \ker(A-2I)$

$$v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$v_1 \in \ker(A-2I)$ & $v_1 \notin \ker((A-2I)^2)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P \quad \& \quad J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$x = Py = c_1 e^{2t} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ t \end{bmatrix}$$

FSM:

$F' = AF$ where $F(0) = I$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{bmatrix}$$

12.2 Non-homogeneous systems

Find the particular solution for the following system.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ 6e^t \end{bmatrix}$$

Solve x_c

$$A-\lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$$

$\lambda = 4$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x_c = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solve x_p

$$x = e^t a$$
$$x' = e^t a$$
$$x' = Ax + f$$
$$\cancel{e^t} a = \cancel{e^t}[A]a + \cancel{e^t}\overbrace{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}^{f}$$
$$a = [A]a + \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$a - [A]a = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$(I - [A])a = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

since $\lambda = 1$ is not an eigenvalue of A, $(I-[A])^{-1}$ exists, so

$$(I-[A])^{-1}(I-[A])a = (I-A)^{-1}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$a = (I-A)^{-1}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} -6 \\ -12 \end{bmatrix}$$

$x = x_c + x_p$

$$x = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \end{bmatrix} e^t$$

Find the general solution of the following system.

$$\vec{x}'(t) = \begin{bmatrix} -2 & 2 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \vec{x}(t).$$

Hint:
$$\underset{A}{\begin{bmatrix} -2 & 2 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}} \underset{J}{\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}^{-1}}$$

$x = Py$

$$y = \underbrace{c_1 e^{-2t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}t\\1\\0\\0\end{bmatrix}}_{\text{Jordan block \#1}} + \underbrace{c_3 e^{-2t}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + c_4 e^{-2t}\begin{bmatrix}0\\0\\t\\1\end{bmatrix}}_{\text{Jordan block \#2}}$$

$$x = c_1 e^{-2t}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix}t\\1\\0\\0\end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + c_4 e^{-2t}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix}0\\0\\t\\1\end{bmatrix}$$

$$\boxed{x = c_1 e^{-2t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}t\\1\\1\\-1\end{bmatrix} + c_3 e^{-2t}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + c_4 e^{-2t}\begin{bmatrix}t\\0\\-t+1\\1\end{bmatrix}}$$