11 Systems of differential equations

11.1 Basics of systems of differential equations

Convert to an equivalent first order system.

$$y'' + 17y' + 3y = e^x$$

$x_1 = y$, $x_2 = y'$ $\Big\}$ $x_2' = y''$, $x_1' = x_2$

$y'' + 17y' + 3y = e^x$

$x_2' + 17x_2 + 3x_1 = e^x$

$\hookrightarrow x_2' = -17x_2 - 3x_1 + e^x$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^x \end{bmatrix}$$

Convert to an equivalent first order system.

$$x'' = x + y$$
$$y'' = 2x - 3y$$

$x'' = x + y$

$y'' = 2x - 3y$

$x_2' = x_1 + x_3$

$x_4' = 2x_1 - 3x_3$

$x_1 = x$ $\quad x_3 = y$

$x_2 = x'$ $\quad x_4 = y'$

$\therefore x_2 = x_1'$ $\quad \therefore x_4 = x_3'$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Rewrite the following first-order system into a second-order equation for $x(t)$.

$$x' = 3x + 2y$$
$$y' = x - y$$

$x' = 3x + 2y \Rightarrow y = \frac{1}{2}x' - \frac{3}{2}x$

$x'' = 3x' + 2y' \Rightarrow y' = \frac{1}{2}x'' - \frac{3}{2}x'$

$y' = x - y$

$\frac{1}{2}x'' - \frac{3}{2}x' = x - \left(\frac{1}{2}x' - \frac{3}{2}x\right)$

$$\boxed{\frac{1}{2}x'' - x' - \frac{5}{2}x = 0}$$

11.2 Constant coefficient systems

Find the general and specific solutions of the following system.

$$x' = 2x - 3y$$
$$y' = -3x + 10y$$

$$x(0) = 1 \quad y(0) = 4$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A - \lambda I = \begin{vmatrix} 2-\lambda & -3 \\ -3 & 10-\lambda \end{vmatrix} = (\lambda^2 - 12\lambda + 11)$$
$$= (\lambda - 11)(\lambda - 1)$$
$$\lambda = 1, 11$$

$\underline{\lambda = 1}$ $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\underline{\lambda = 11}$ $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{11t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = c_1 e^{0} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{0} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$-1 = 3c_1 + c_2$$
$$4 = c_1 - 3c_2$$

$$c_1 = \frac{1}{10} \quad c_2 = \frac{-13}{10}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10} e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{-13}{10} e^{11t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$$

Find the general and specific solutions of the following equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$A - \lambda I = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

$\underline{\lambda = i}$

$$A - iI = \left[\begin{array}{cc|c} -i & 1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Plug into $e^{at}(\cos(bt) + i\sin(bt))\begin{bmatrix} v \end{bmatrix}$

$$= e^{0t}(\cos(t) + i\sin(t))\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) + i\sin(t) \\ i\cos(t) + -\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + i \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore c_1 = 6 \quad c_2 = 3$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + 3 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}$$