10 Physical applications and Cauchy-Euler equations

10.1 Springs and oscillations

Are these systems underdamped, critically damped, or overdamped?

- $y'' + 4y' + 10y = 0$

$b^2 \bigcirc 4km$	
$<$	$\Rightarrow$ UNDERDAMPED
$=$	$\Rightarrow$ CRITICALLY DAMPED
$>$	$\Rightarrow$ OVERDAMPED

4^2 ___ $4(10)(1)$

$16 < 40$

UNDERDAMPED

- $y'' + 10y' + 4y = 0$

$b^2 \bigcirc 4km$	
$<$	$\Rightarrow$ UNDERDAMPED
$=$	$\Rightarrow$ CRITICALLY DAMPED
$>$	$\Rightarrow$ OVERDAMPED

10^2 ___ $4(4)(1)$

$100 > 16$

OVERDAMPED

- $2y'' + 4y' + 2y = 0$

$b^2 \bigcirc 4km$	
$<$	$\Rightarrow$ UNDERDAMPED
$=$	$\Rightarrow$ CRITICALLY DAMPED
$>$	$\Rightarrow$ OVERDAMPED

4^2 ___ $4(2)(2)$

$16 = 16$

CRITICALLY DAMPED

Do the following systems exhibit modulation, transience, or resonance?

- $y'' + 4y' + 20y = 7\cos 3t$

DAMPING! transience

- $y'' + 81y = 12\sin 9t$

NO DAMPING... modulation or resonance

$\omega = \sqrt{\frac{k}{m}}$

$\omega = \sqrt{\frac{81}{1}} = \sqrt{81} = 9$

$\alpha = 9$

$\alpha = \omega \Rightarrow$ resonance

- $y'' + 81y = 12\sin 11t$

NO DAMPING... modulation or resonance

$\omega = \sqrt{\frac{k}{m}}$

$\omega = \sqrt{\frac{81}{1}} = \sqrt{81} = 9$

$\alpha = 11$

$\alpha \neq \omega \Rightarrow$ modulation

10.2 Cauchy-Euler equations

Solve the following Cauchy-Euler equation with initial values:

$$x^2y'' - 3xy' + 3y = 12x^4$$

$$y(0) = 1 \quad \text{and} \quad y'(0) = 2$$

#1: Solve y_c

$x^2y'' - 3xy' + 3y = 0$

$[y = x^p,\ y' = px^{p-1},\ y'' = p(p-1)x^{p-2}]$

$x^2[p(p-1)x^{p-2}] - 3x[px^{p-1}] + 3[x^p] = 0$

$p(p-1)\cancel{x^2}x^p\cancel{x^{-2}} - 3p\cancel{x}x^p\cancel{x^{-1}} + 3x^p = 0$

$x^p[\underbrace{p(p-1) - 3p + 3}] = 0$

$p^2 - 4p + 3 = 0 = (p-3)(p-1) \therefore p = 1, 3$

$[y = c_1x^{p_1} + c_2x^{p_2}]$

$y_c = c_1x + c_2x^3$

#2: Solve y_p

$F(t) = 12x^4,\ \alpha = Ax^4$

$\alpha = Ax^4;\ \alpha' = 4Ax^3;\ \alpha'' = 12Ax^2$

$x^2(12Ax^2) - 3x(4Ax^3) + 3(Ax^4) = 12x^4$

$\cancel{12Ax^4 - 12Ax^4} + 3Ax^4 = 12x^4$

$A = 4 \therefore y_p = 4x^4$

$y = y_c + y_p = \boxed{c_1x + c_2x^3 + 4x^4 = y}$

#3: Solve IVP $y(0) = 1,\ y'(0) = 2$