Week 8

Interactions between linear transformations and inner products, symmetric matrices, Singular Value Decomposition

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8.1: Further Transformations

The concepts of “Inner Products and Linear Transformations,” “Self-Adjoint Transformations,” and “Symmetric Matrices” are key to understanding the geometric structure of vector spaces in linear algebra.

- Orthogonal transformations: $\langle v, w \rangle = \langle T(v), T(w) \rangle$
  - Orthogonal matrices
- Self-Adjoint transformations: $\langle T(v), w \rangle = \langle v, T(w) \rangle$
  - If v belongs to the nullspace and w belongs to the image, then $\langle v,w \rangle = 0$
  - The nullspace and the image are complementary orthogonal subspaces of $\mathbf{R}^n$
  - $D = P^T A P$
- Symmetric matrices: always diagonalizable with real eigenvalues with orthogonal eigenvectors
  - $D = P^{-1} A P = P^T A P$

8.1: Spectral Theorem and Decomposition

The concepts of "Inner Products and Linear Transformations," "Self-Adjoint Transformations," and "Symmetric Matrices" are key to understanding the geometric structure of vector spaces in linear algebra.

- Spectral Theorem: for nxn symmetric matrix, A
  - A has n real λs (with according multiplicities)
  - # linearly independent eigenvectors of each λ = multiplicity of λ
  - Eigenvectors with distinct λs are orthogonal
  - A is orthogonally diagonalizable st $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{T}\mathbf{AP}$
- Spectral Decomposition: $\mathbf{A} = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^T + \ldots + \lambda_n \mathbf{v}_n \mathbf{v}_n^T$
  - $\mathbf{tr(A^TA)} = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2$

8.1.1: Symmetric Matrix Example

Orthogonally diagonalize the symmetric matrix and show its spectral decomposition.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

8.1.2: Symmetric Matrix Example

Orthogonally diagonalize the symmetric matrix

$$A = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 3 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

8.2: Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD) and Principal Component Analysis (PCA) are two powerful techniques in linear algebra and data analysis that are widely used for dimensionality reduction, data compression, and uncovering the underlying structure in datasets.

- Singular Value Decomposition: matrix factorization
  - Singular values(σ): square roots of eigenvalues of AA^T
  - A^TA: symmetric, tells us the λs are real and the singular values are ≥ 0
- $\mathbf{A} = \mathbf{Q\Sigma P^{-1}}$ and $\mathbf{A^T} = \mathbf{P\Sigma^T Q^T}$
  - Singular values of A are the same as A^T

Finding SVD for A_{mxn}

where m>n

1. Find singular values of A by finding σ's = sqrt(λ) (for A^TA) to make Σ
2. Calculate **right singular vectors** to make columns of **P** by finding eigenvectors of A^TA(must be normalized)
3. To find first n columns of Q, find AP & normalize, for the other (m-n) columns, orthogonal vectors in nullspace(A^T) that are orthogonal to first n column vectors (**left singular vectors**)

where m<n

1. Find SVD of A^T (as for m>n) to get $\mathbf{A^T=P\Sigma^T Q^T}$ and then transpose to get $\mathbf{A=Q\Sigma P^T}$

8.2: Principal Component Analysis (PCA)

Singular Value Decomposition (SVD) and Principal Component Analysis (PCA) are two powerful techniques in linear algebra and data analysis that are widely used for dimensionality reduction, data compression, and uncovering the underlying structure in datasets.

- Principal Component Analysis: interpreting data sets with higher dimensions
  - $A_{mxn} = [\ x_1 \mid \ldots \mid x_n\ ]$ where $n >> m$
  - Mean = $(1/n)\ \Sigma\ x_j$
  - Centered data: $B_{mxn} = [\ \hat{x}_1 \mid \ldots \mid \hat{x}_n\ ]$ where $\hat{x}_j = x_j - m$
  - Covariance matrix: $S = (1/(n-1))BB^T$
    - Variance: diagonal of S
    - Total variance: tr(S)
  - PCA finds the fewest linear combinations of x's to account for the variation of the observations
  - Principal components are the **left singular vectors** of B

8.2.1: SVD Example

Perform the singular value decomposition on this matrix, A a.k.a. Find Σ, P, Q

$$A = \begin{bmatrix} -2 & 5 & 4 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$