Week 13

Geometry of systems of differential equations

MATH2400 | Ryann Joseph | ryannj@seas.upenn.edu | Fall 2024

13.1: Geometry of Systems of Diff-Eqs

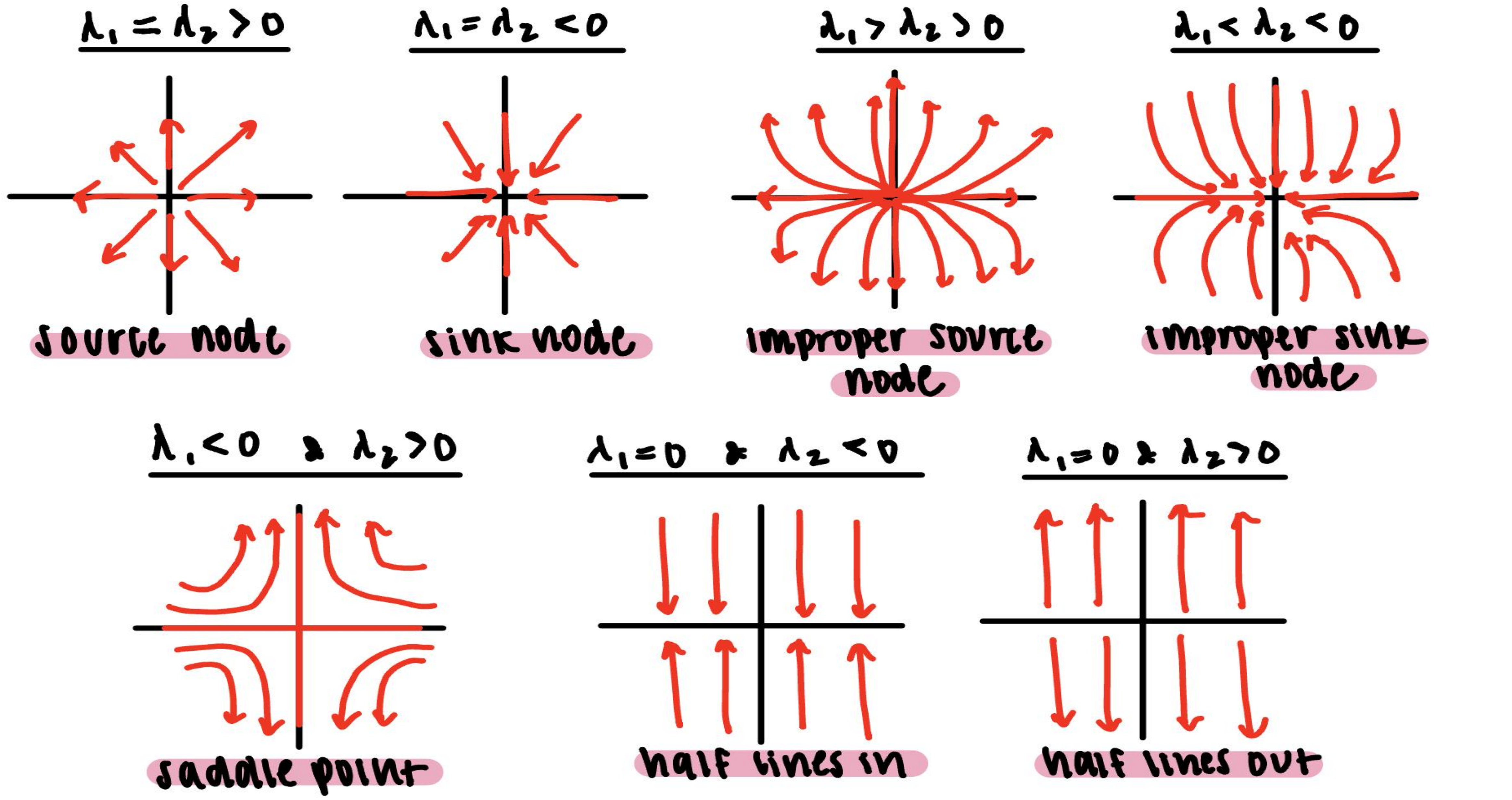
The geometry of systems of differential equations involves visualizing and understanding the behavior of solutions in the phase plane, which provides insights into the qualitative behavior of systems.

- **Critical point**: when $F'(x_0) = 0$
  - **Stable**: $F'(x_0) < 0$
    - To the left in 1D
  - **Unstable**: $F'(x_0) > 0$
    - To the right in 1D

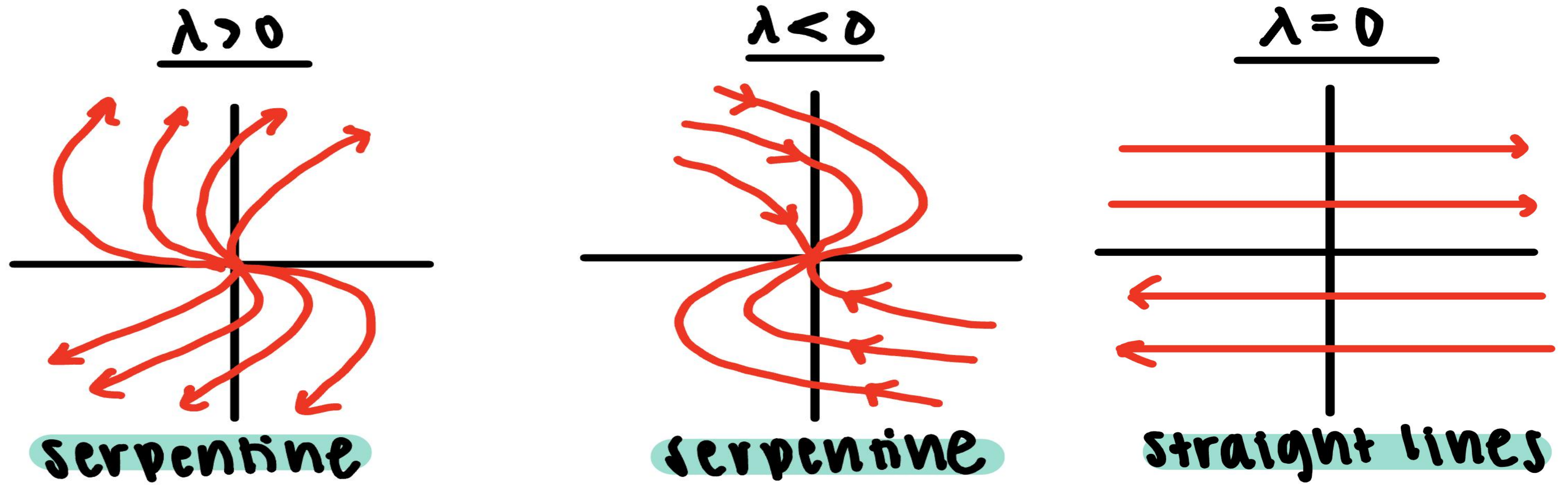
$x' = Ax$	
Eigenvalues	**Geometry**
Diagonalizable	
Equal & real	Source/sink node
Distinct & real	Improper source/sink node
Opposite signs & real	Saddle point
One is zero, one is nonzero	Comb
Complex, $a \neq 0$	Spiral source/sink
Complex, $a = 0$	Center
Non-diagonalizable	
Nonzero	Defective source/sink node
Zero	Parallel lines

13.1: Phase Portrait Criteria

Real Eigenvalues, Diagonal

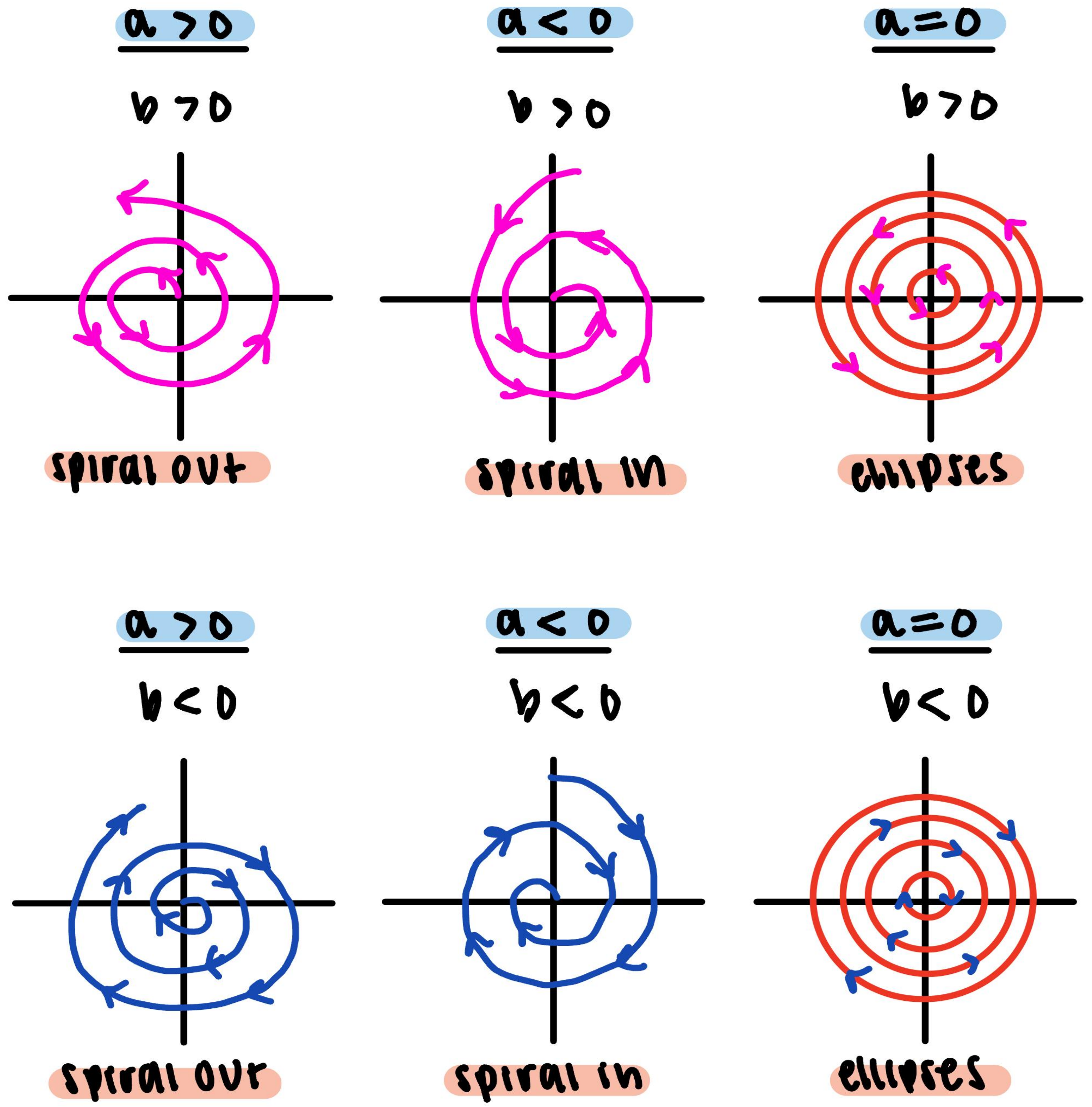


Real Eigenvalues, Non-Diagonal



Complex Eigenvalues, Diagonal

$\lambda = a \pm ib$ $b > 0$: counterclockwise $b < 0$: clockwise



13.1: Graphing Phase Portraits (1 CP)

Critical points and eigenvalues guide phase portrait construction, revealing system stability and dynamics through eigenvector-aligned trajectories.

- Find critical point by setting **x' = 0 and y' = 0**, this is the center of your portrait
- Solve **x'=Ax** to find the eigenvalues and eigenvectors
- Determine the **type of critical point** based on the eigenvalues (real/complex, positive/negative, equal/distinct)
- Draw the eigenvectors centered at the critical point
- Draw integral curves following the eigenvectors' trajectories (do **not** intersect the eigenvectors)
  - Parallel to eigenvector with eigenvalue of the **greatest value**
  - Tangent to eigenvector with eigenvalue of the **least value**
  - **Make sure to show directionality (arrows in / arrows out)**

13.1.1: 1D Critical Point Example

Identify the critical points of the equation.

$$x' = 2x - x^2$$

13.1.2: 2D Critical Point Example

Consider the linear system $x' = Ax$, where $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Where is the critical point of this system and what kind of critical point is it?

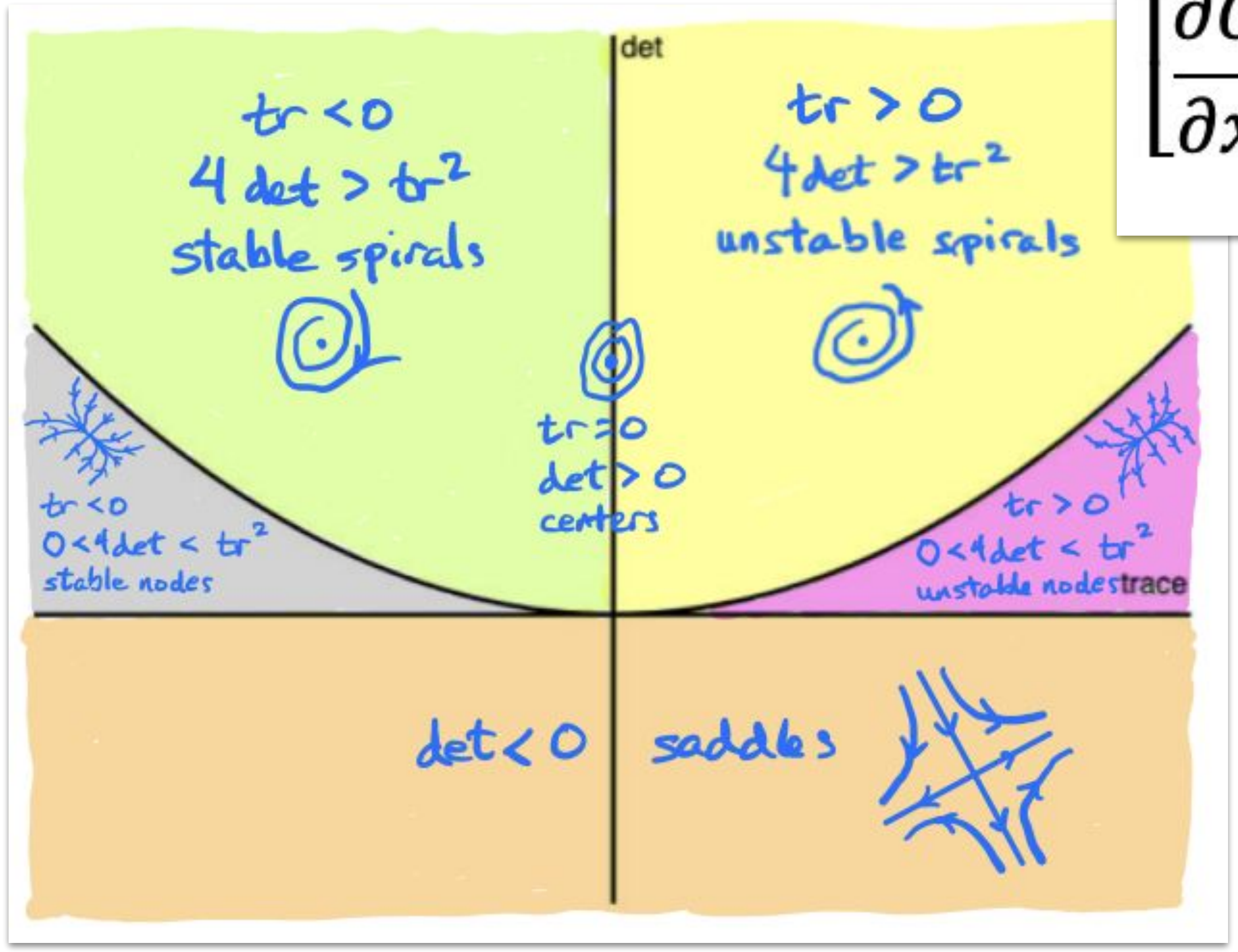
Sketch the phase portrait for this system. Pay attention to the direction of the trajectories as $t \to \pm\infty$

13.2: Nonlinear Autonomous Systems

Nonlinear autonomous systems are systems of differential equations where the variables interact in a non-linear way and do not explicitly depend on time.

- **Jacobian**: matrix of partial derivatives
  - We can evaluate the Jacobian at the critical points to determine what type of point it is

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$



Jacobian	
Eigenvalues	**Geometry**
Real & (+)	Unstable node
Real & (-)	Stable node
Real & (+/-)	Saddle point
Complex, (+) real part	Unstable spiral
Complex, (-) real part	Stable spiral
Complex, a = 0	Center or spiral

13.2: Graphing Phase Portraits (2 CPs)

Jacobian matrix analysis at critical points defines system behavior, guiding phase portrait creation with eigenvalue-based trajectory alignment.

- Set **dx/dt** = **F** and **dy/dt** = **G** (this is arbitrary)
- Solve for the critical points
  - Each ordered pair is where **x sets F or G = 0**, **y sets G or F = 0**
- Find the Jacobian matrix

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$

- Plug each critical point ordered pair into the Jacobian matrix
- Determine the **type of critical point** based on the eigenvalues (real/complex, positive/negative, equal/distinct) for **each plugged in Jacobian matrix**
- Draw the eigenvectors centered at the **respective critical point**
- Draw integral curves following the eigenvectors' trajectories (do **not** intersect the eigenvectors)
  - <u>Parallel</u> to eigenvector with eigenvalue of the **greatest** **<u>value</u>**
  - <u>Tangent</u> to eigenvector with eigenvalue of the **least** **<u>value</u>**
  - **Make sure to show directionality (arrows in / arrows out)**

13.2.1: 2D Critical Point Example

Find and classify all of the critical (equilibrium) points of the nonlinear system.

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x^3 - x - y$$

13.3: Physical applications

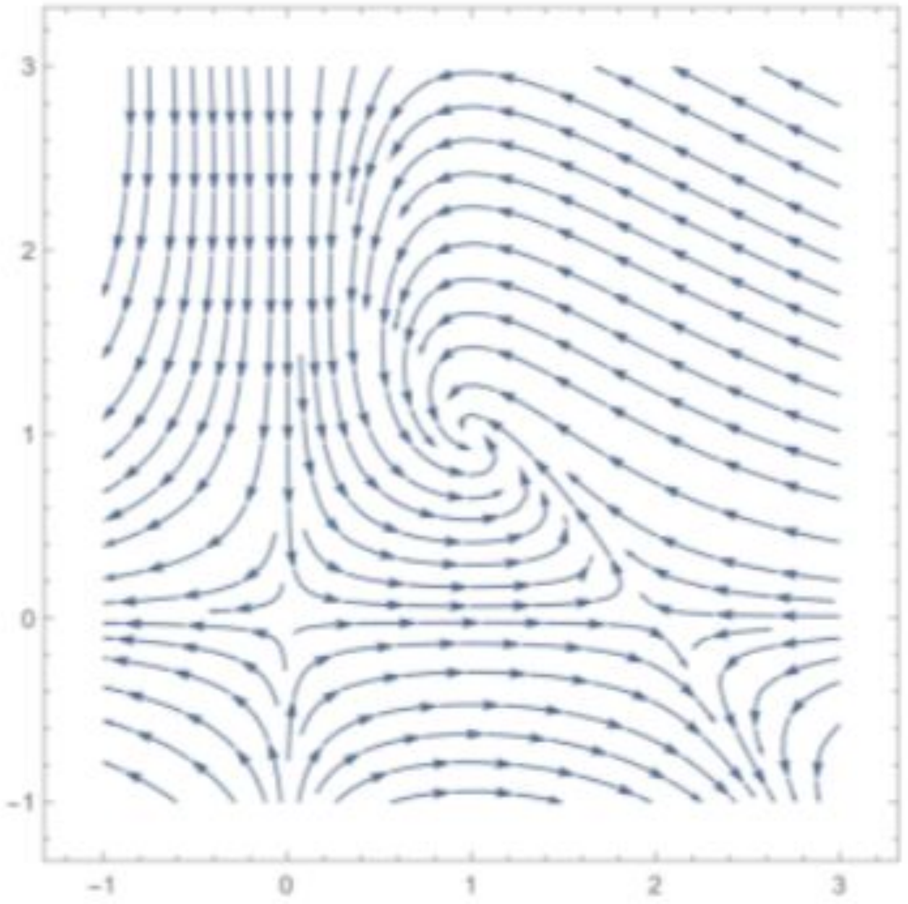
Physical applications of differential equations showcase how these mathematical tools can model and solve real-world problems across various scientific disciplines.

Dual Population Ecosystem

We will encounter the following setup: we consider an ecosystem where there are bugs (whose population is given by $x(t)$) and birds (whose population is given by $y(t)$). Birds eat bugs, so the two populations are connected by the following non-linear system:

$$x' = (2 - x - y)x,$$
$$y' = (x - 1)y.$$

Here is the phase portrait of the system:



Pesticide Control

Now suppose you introduce pesticide into the ecosystem, which decreases the population growth of both bugs and birds. The new system takes the form:

$$x' = (2 - x - y - a)x,$$
$$y' = (x - 1 - b)y,$$

where a, b are certain positive constants that depend on the pesticide. Find the new critical point for which both x and y are positive.