Week 11

Systems of differential equations

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11.1: Converting Systems to Matrix Systems

Systems of differential equations involve multiple interrelated differential equations that describe how several variables change with respect to one another over time.

Systems of 1st order differential equations

$$x_1' = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + f_1$$

$$x_n' = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + f_n$$

$$\frac{dx}{dt} = Ax + f$$

$$\frac{dx}{dt}\begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} \begin{bmatrix} f_1 \\ ... \\ f_n \end{bmatrix}$$

Homogeneous: if $f = 0$
Non-homogeneous: if $f \neq 0$

Converting from 2nd order to 1st order systems

$$x" + p(t)x' + q(t)x = f(t)$$

- $x' = y$
- $y' = -q(t)x - p(t)y + f$

11.1.1: 1st Order, Single Variable Example

Convert to an equivalent first order system.

$$y'' + 17y' + 3y = e^x$$

11.1.2: 1st Order, Multi-Variable Example

Convert to an equivalent first order system.

$x" = x + y$
$y" = 2x - 3y$

11.1.3: 1st to 2nd Order Example

Rewrite the following first-order system into a second-order equation for $x(t)$.

$$x' = 3x + 2y$$
$$y' = x - y$$

11.2: Systems with Real Eigenvalues

Constant coefficient systems refer to systems of differential equations where the coefficients are constant, making them easier to analyze and solve.

Real eigenvalues

$$x' = 5x + 3y$$
$$y' = 3x + 5y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Solve matrix, A, to get

$\lambda = 8$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = 2$ with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The solution is then

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

11.2: Systems with Complex Eigenvalues

Constant coefficient systems refer to systems of differential equations where the coefficients are constant, making them easier to analyze and solve.

Complex eigenvalues

$x' = 3x + 4y$
$y' = -4x + 3y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Solve matrix, A, to get $\lambda = 3 \pm 4i$, then we take the **positive** eigenvalue, $\lambda = 3 + 4i$ and solve for the eigenvector to get $\begin{bmatrix} 1 \\ i \end{bmatrix}$

Now, we plug this into the general solution form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{at}(\cos(bt) + i\sin(bt)) \begin{bmatrix} v \end{bmatrix}$$

where $\lambda = a \pm bi$.

So, for this example, we plug in to get

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t}(\cos(4t) + i\sin(4t)) \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{3t}\cos 4t \\ -e^{3t}\sin 4t \end{bmatrix} + i \begin{bmatrix} e^{3t}\sin 4t \\ e^{3t}\cos 4t \end{bmatrix}$$

The i gets absorbed into the constants, and our final solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} e^{3t}\cos 4t \\ -e^{3t}\sin 4t \end{bmatrix} + c_2 \begin{bmatrix} e^{3t}\sin 4t \\ e^{3t}\cos 4t \end{bmatrix}$$

11.2.1: Real Solution Example

Find the general and specific solutions of the following system.

$$x' = 2x - 3y \qquad x(0) = 1$$
$$y' = -3x + 10y \qquad y(0) = 4$$

11.2.2: Complex Solution Example

Find the general and specific solutions of the following equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$