Week 10

Physical applications and Cauchy-Euler equations

MATH2400 | Ryann Joseph | ryannj@seas.upenn.edu | Fall 2024

10.1: Spring Damping

Springs and oscillations are classic examples of physical systems that can be modeled using linear differential equations, particularly in the context of harmonic motion.

Damping

Free Oscillation		Forced Oscillation	
Undamped	**Damped**	**Undamped**	**Damped**
$y'' + (k/m)y = 0$	$my'' + by' + ky = 0$	$my'' + ky = F\cos(\alpha t)$	$my'' + by' + ky = F\cos(\alpha t)$
No friction, no force	Friction, no force	No friction, force	Friction, force

$b^2(\ )4km$	Type of damping
$b^2 < 4km$	Underdamped
$b^2 = 4km$	Critically damped
$b^2 > 4km$	Overdamped

10.1: Forced Oscillations

Springs and oscillations are classic examples of physical systems that can be modeled using linear differential equations, particularly in the context of harmonic motion.

Forced Oscillations

Forced Oscillations	
Damped ($b \neq 0$)	Transience
Undamped ($b = 0$)	If α in $F \cos \alpha t$ is **equal** to ω → Resonance
	If α in $F \cos \alpha t$ is **not equal** to ω → Modulation

$$\text{Undamped: } \omega = \sqrt{\frac{k}{m}}$$

$$\text{Damped: } \omega = \frac{\sqrt{4km - b^2}}{2m}$$

10.1.1: Damping Example

Are these systems underdamped, critically damped, or overdamped?

- $y'' + 4y' + 10y = 0$
- $y'' + 10y' + 4y = 0$
- $2y'' + 4y' + 2y = 0$

10.1.2: Wave form Example

Do the following systems exhibit modulation, transience, or resonance?

- $y'' + 4y' + 20y = 7\cos 3t$
- $y'' + 81y = 12\sin 9t$
- $y'' + 81y = 12\sin 11t$

10.2: Cauchy Euler Equations

Cauchy-Euler equations are a specific type of linear differential equation characterized by variable coefficients that follow a power law pattern.

Cauchy Euler Equations

$Ax^2y" + Bxy' + Cy = F(x)$

Homogeneous: $F(t) = 0$

Nonhomogeneous: $F(t) \neq 0$

$y = y_c + y_p$

To solve: $\boldsymbol{y = x^p}$

Solving y_c

p_1, p_2: real, distinct	$y = c_1x^{p_1} + c_2x^{p_2}$
$p_1, p_2 = a \pm bi$: complex, distinct	$y = c_1x^a \cos(b \ln x) + c_2x^a \sin(b \ln x)$
$p_1 = p_2$: real, repeated	$y = c_1x^p + c_2x^p \ln(x)$

10.2.1: Cauchy Euler Example

Solve the following Cauchy-Euler equation with initial values:

$$x^2y'' - 3xy' + 3y = 12x^4$$

$$y(0) = 1 \quad \text{and} \quad y'(0) = 2$$