8 Interactions between linear transformations and inner products; symmetric matrices; Singular Value Decomposition

8.1 Inner products and linear transformations, self-adjoint transformations and symmetric matrices

Orthogonally diagonalize the symmetric matrix and show its spectral decomposition.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

1) find $\det(A-\lambda I)$ / eigenvalues

$$A-\lambda I = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3+4\lambda^2+\lambda-4 = (1-\lambda)(1+\lambda)(\lambda-4)$$

$\lambda = 1, -1, 4$

2) find eigenvectors

$$\underline{\lambda=-1}: \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{\lambda=1}: \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \underline{\lambda=4}: \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3) form D

$$\boxed{D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

4) form P'

5) form P (normalized)

$$P' = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{needs to be normalized}} \boxed{\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{bmatrix} = P}$$

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T$$

spectral decomposition: $\boxed{A = 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} + -\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}}$

Orthogonally diagonalize the symmetric matrix

$$A = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 3 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

1) find $\det(A-\lambda I)$ / find λ's

$$A-\lambda I = \begin{vmatrix} 6-\lambda & 2 & -2 \\ 2 & 3-\lambda & 4 \\ -2 & 4 & 3-\lambda \end{vmatrix} \Rightarrow \chi(\lambda) = (6-\lambda)\left[(3-\lambda)(3-\lambda)-16\right] - 2\left[2(3-\lambda)-(-2)(4)\right] + -2\left[(2)(4)-(-2)(3-\lambda)\right]$$

$$\vdots$$

$$= \lambda^3 - 12\lambda^2 + 21\lambda + 98$$

$$= (\lambda+2)(\lambda-7)^2 \qquad \lambda = -2, 7$$

2) find eigenvectors

$\underline{\lambda=-2}$

$$\left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 2 & 5 & 4 & 0 \\ -2 & 4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Null}(A+2I) = \left\{\begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}\right\} \xrightarrow{\text{normalize}} \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$\underline{\lambda=7}$

$$\left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \\ -2 & 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Null}(A-7I) = \left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\} \xrightarrow{\text{normalize}} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

3) find orthogonal eigenvectors

As we see, we have:

$$\underset{v_1}{\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}}$$

$$\langle v_1, v_2 \rangle = \frac{1}{3}\left(\frac{2}{\sqrt{5}}\right) + \frac{-2}{3}\left(\frac{1}{\sqrt{5}}\right) + \frac{2}{3}(0) = 0$$

$$\langle v_1, v_3 \rangle = \frac{1}{3}\left(\frac{-2}{\sqrt{5}}\right) + \frac{-2}{3}(0) + \frac{2}{3}\left(\frac{1}{\sqrt{5}}\right) = 0$$

$$\langle v_2, v_3 \rangle = \frac{2}{\sqrt{5}}\left(\frac{-2}{\sqrt{5}}\right) + \frac{1}{\sqrt{5}}(0) + 0(0) = -\frac{4}{5} \neq 0$$

must find orthogonal vectors

Let's make v_3', all of v_3 that is orthogonal to v_2

$$v_3' = v_3 - \langle v_3, v_2 \rangle v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} + \frac{4}{5}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} + \begin{bmatrix} 8/5\sqrt{5} \\ 4/5\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5\sqrt{5} \\ 4/5\sqrt{5} \\ 5/5\sqrt{5} \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} -2/3\sqrt{5} \\ 4/3\sqrt{5} \\ 5/3\sqrt{5} \end{bmatrix}$$

4) form D, P

$$\boxed{D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}}$$

$$\boxed{P = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}}$$

8.2 Singular value decomposition and Principal component analysis

Perform the singular value decomposition on this matrix, A a.k.a. Find Σ, P, Q

$$A = \begin{bmatrix} -2 & 5 & 4 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

we need $m > n$, so we perform SVD on $B = A^T$

$$B = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ 4 & 0 \\ 2 & 1 \end{bmatrix}$$

1) Find singular values, σ

$$B^TB = \begin{bmatrix} -2 & 5 & 4 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 49 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\sigma = \sqrt{49}, \sqrt{2} = 7, \sqrt{2}$$

2) Form Σ

Σ is the same size of B

$$\Sigma = \begin{bmatrix} 7 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3) Find right singular vectors (in P, belong to $\text{null}(B^TB)$)

(must be normalized)

$\lambda = 49$

$$(B^TB - 49I) = \left[\begin{array}{cc|c} 49-49 & 0 & 0 \\ 0 & 2-49 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$$(B^TB - 2I) = \left[\begin{array}{cc|c} 49-2 & 0 & 0 \\ 0 & 2-2 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4) Find n left singular vectors (belong to $\text{null}(B^T)$)

(must be normalized)

$$Bp_1 = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 4 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2/7 \\ 5/7 \\ 4/7 \\ 2/7 \end{bmatrix}$$

$$Bp_2 = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

5) Find remaining $m-n$ vectors in $\text{null}(B^T)$

$$m - n = 4 - 2 = 2$$

$$B^T = \begin{bmatrix} -2 & 5 & 4 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}(B^T)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0.8 & 0.8 & 0 \end{array}\right] \Rightarrow \text{kernel}(B^T) = \left\{ \begin{bmatrix} 0 \\ -0.8 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -0.8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

normalize

$$\left\{ \begin{bmatrix} 0 \\ -4/\sqrt{41} \\ 5/\sqrt{41} \\ 0 \end{bmatrix}, \begin{bmatrix} -5/\sqrt{66} \\ -4/\sqrt{66} \\ 0 \\ 5/\sqrt{66} \end{bmatrix} \right\}$$

* must perform gram schmidt if not orthogonal to n vectors *

6) Form Q

$$Q = \begin{bmatrix} -2/7 & 1/\sqrt{2} & 0 & -5/\sqrt{66} \\ 5/7 & 0 & -4/\sqrt{41} & -4/\sqrt{66} \\ 4/7 & 0 & 5/\sqrt{41} & 0 \\ 2/7 & 1/\sqrt{2} & 0 & 5/\sqrt{66} \end{bmatrix}$$

7) Find Σ^T, P^T, and Q^T (since we solved for $B = A^T$, not A)

$$\Sigma = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} -2/7 & 5/7 & 4/7 & 2/7 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & -4/\sqrt{41} & 5/\sqrt{41} & 0 \\ -5/\sqrt{66} & -4/\sqrt{66} & 0 & 5/\sqrt{66} \end{bmatrix}$$