# 7 More on eigenvalues and eigenvectors; Jordan Canonical form; inner products

## 7.1 Diagonalization and what can go wrong

Let $A$ be a $7 \times 7$-matrix. Suppose its characteristic polynomial is given by $(\lambda - 2)^7$. You are told that the nullity of $A - 2I_7$ is 3, the nullity of $(A - 2I_7)^2$ is 5, and the nullity of $(A - 2I_7)^3$ is 7. Therefore $A$ is not diagonalizable.

- Explain how many Jordan blocks associated to $A$ there are.

$\chi(\lambda) = (\lambda - 2)^{7}$

Nullity$(A - 2I) = 3$

7: # $\lambda = 2$ along diagonal

3: # Jordan blocks

$\boxed{\text{3 Jordan blocks}}$

- What is the size of the largest Jordan block associated to $A$? Why?

Nullity$(A - 2I)^2 = 5$

Nullity$(A - 2I)^3 = 7$

power when Nullity = alg. mult.

3: largest J block is 3x3

$\boxed{\text{3x3 is largest Jordan block}}$

- Provide the Jordan canonical form of $A$ and explain how you got it.

*Find J*

largest block is 3, we have 7, 3 blocks:

could be:

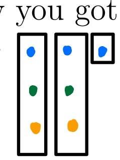
3, 3, 1

3, 2, 2

Nullity$(A-2I) = 3$

Nullity$(A-2I)^2 -$ Nullity$(A-2I) = 5 - 3 = 2$

Nullity$(A-2I)^3 -$ Nullity$(A-2I)^2 = 7 - 5 = 2$



$\Rightarrow$ 3x3, 3x3, 1x1

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Find the Jordan canonical form, $J = P^{-1}AP$, for the following matrix

$$A = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix}$$

**1) FIND λs**

$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 4 & 5-\lambda \end{bmatrix} \Rightarrow \det = (1-\lambda)(5-\lambda) + 4 = 0$

$= \lambda^2 - 6\lambda + 9$

$= (\lambda - 3)^2$

$\lambda = 3$

mult. $= 2$

**2) FIND A-3I**

$A - 3I = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \xrightarrow{R2 = R2 + 2R1} \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$ rank = 1, Nullity = 1 < 2 = multiplicity

kernel $= \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

**3) FIND $(A-3I)^2$**

$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ rank = 0, Nullity = 2 = multiplicity

**4) FIND $v_2$**

$v_2 \in \ker((A-3I)^2)$ & $v_2 \notin \ker(A-3I)$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ kernel $= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

**5) FIND $v_1$**

$v_1 \in \ker(A-3I)$ & $v_1 \notin \ker((A-3I)^2)$

$\ker(A-3I) = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ & $\ker(A-3I)^2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$v_1 = (A-3I)v_2 = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

**6) FORM P and J**

$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} = P$

$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = J$

$J = P^{-1}AP$

$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix}$

Find the Jordan canonical form, $J = P^{-1}AP$, for the following matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

**1) FIND λ's**

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 0 & 0 & -\lambda \end{bmatrix} \Rightarrow \det \Rightarrow = -\lambda(\lambda^2 - 0) - 0(-\lambda - 0) + 1(0 - 0)$$

$$= -\lambda^3 = 0 \qquad \lambda = 0,\ \text{mult.} = 3$$

**2) FIND $A - 0I \mid 0$**

$$A - 0I = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{array}{l} \text{rank} = 2 \\ \text{nullity} = 1 \end{array} \rightarrow \text{kernel} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

**3) FIND $(A - 0I)^2 \mid 0$**

$$(A - 0I)^2 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{array}{l} \text{rank} = 1 \\ \text{nullity} = 2 \end{array} \rightarrow \text{kernel} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

**4) FIND $(A - 0I)^3 \mid 0$**

$$(A - 0I)^3 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{array}{l} \text{rank} = 0 \\ \text{nullity} = 3 \end{array} \rightarrow \text{kernel} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{generalized eigenvector}} \right\}$$

**5) FIND $v_1, v_2, v_3$**

$v_1 \in \ker(A - 0I)$
$v_1 \notin \ker((A - 0I)^2)$
$v_1 \notin \ker((A - 0I^3))$

$$v_1 = (A - 0I)^2 v_g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$v_2 \in \ker((A - 0I)^2)$
$v_2 \notin \ker(A - 0I)$
$v_2 \notin \ker((A - 0I^3))$

$$v_2 = (A - 0I) v_g = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$v_3 \in \ker((A - 0I^3))$
$v_3 \notin \ker(A - 0I)$
$v_3 \notin \ker((A - 0I)^2)$

$$v_3 = v_g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

**6) FORM P and J**

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P}$$

$$\boxed{J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

## 7.2 Inner products and orthonormal bases

Using the Gram-Schmidt process, find an orthonormal basis for the column space for this matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Basis}(A) = \left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

1. Set $e_1 = \dfrac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

2. Project $v_2$

$$w_2 = v_2 - \langle v_2, e_1 \rangle e_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$$

3. normalize $w_2$

$$e_2 = \frac{w_2}{\|w_2\|} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2\sqrt{6} \\ 0 \end{bmatrix}$$

4. project $w_3$

$$w_3 = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - 0 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} + \frac{2}{\sqrt{6}} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2\sqrt{6} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

5. Normalize $w_3$

$$e_3 = \frac{w_3}{\|w_3\|} = \sqrt{\frac{3}{4}} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -3/\sqrt{12} \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -3/\sqrt{12} \end{bmatrix} \right\}$$

$$\text{Let } S \text{ be the subspace of } \mathbb{R}^4 \text{ spanned by } \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Express the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 4 \end{bmatrix}$ as a sum of two vectors $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in S$ and $\mathbf{y}$ is perpendicular to $S$.

we need x to be in the subspace S & we need y to be perpendicular/orthogonal to x, so we need to get the component of v in S by projecting v onto S.

$S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ we need the orthogonal basis for S $\longrightarrow$ let $s_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ & $s_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we project $s_1$ onto $s_2$ to get $s_3$ $\to$ $s_3 = s_2 - \frac{\langle s_1, s_2 \rangle}{\langle s_1, s_1 \rangle} s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = s_3$

Now that we have $S = \{s_1, s_3\}$, an orthogonal basis for S. We want to project v onto S to find the component of v within S, which is x.

$x = \text{proj}_S v = \frac{\langle v, s_1 \rangle}{\langle s_1, s_1 \rangle} s_1 + \frac{\langle v, s_3 \rangle}{\langle s_3, s_3 \rangle} s_3 = \frac{8}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{6}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix} = x$

we know $x + y = v$, so $y = v - x = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} = y$

$\begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 4 \end{bmatrix}$

- Find a basis for the orthogonal complement $S^\perp$ of $S$.

$S^\perp$ is the kernel of the transpose of S $\to \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$ rank = 2, nullity = 2 $\to$ ker = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$