Week 6

Eigenvalues and eigenvectors, diagonalization

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6.1: More on Linear Transformations

The distinction between "Linear Transformations from V to W" and "Linear Transformations from V to V" highlights different aspects of how vector spaces can interact under linear mappings.

- **T: V → W**
  - dim(V) = n & dim(W) = m & rank(T) = r $\Rightarrow 0 \leq r \leq \min(m,n)$ & Nullity(T) = n - r
  - $N = Q^{-1}MP$
- **T: V → V**
- Want T to be as simple as possible given V and W
- Given some M st T: v → Mv, we want to find T in the simplest terms given v
  - $D = P^{-1}MP$

6.1.1: V→W Transformation Example

Find Q, P, N such that $N = Q^{-1}MP$

$$M = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 5 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

6.2: Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are fundamental concepts in linear algebra that reveal important properties of linear transformations and matrices.

In short, when we have a square $n \times n$ matrix, A, we can multiply this matrix by any $n \times 1$ vector, v, to get the resultant $n \times 1$ vector, Av. However, sometimes, we can interchange this matrix multiplication with scalar multiplication by a scalar λ, which is more ideal for us and for computers.

This means that $Av = \lambda v$.

The scalar, λ, that solves this is called an **eigenvalue** of A.

The vector, v, that solves this is called an **eigenvector** of A.

To get our **eigenvalues**, λ, we must solve how $(A - \lambda I) = 0$. In this instance, 0 is a scalar, not a vector, so what we are really looking for is $Det(A - \lambda I) = 0$. This expression is called the **characteristic polynomial**, usually denoted $\chi(\lambda)$.

6.1: Diagonalization

The distinction between “Linear Transformations from V to W” and “Linear Transformations from V to V” highlights different aspects of how vector spaces can interact under linear mappings.

- Diagonalizability: We can find $\lambda_1,\ldots,\lambda_n$ for a given M st $D = P^{-1}MP$

Eigenvectors with different eigenvalues are linearly independent
$\lambda = 0$ can exist if $\lambda = 0$, A^{-1} does not exist
$Det(A) = Det(D)$
$trace(A) = trace(D) = \Sigma\lambda$

$$e^{tr(D)} = e^{tr(M)}$$

$$e^{M} = Pe^{D}P^{-1}$$

6.2.1: Diagonalization of 3x3 Example

Find P, D such that $D = P^{-1}AP$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$