Week 12

More complicated systems of differential equations

MATH2400 | Ryann Joseph | ryannj@seas.upenn.edu | Fall 2024

12.1: Non-Diagonalizable Coefficient Matrices

Derivation of **x=Py**	Solving for y

$$x' = Ax$$
$$A = PJP^{-1}$$
$$x' = PJP^{-1}x$$
$$\text{Let } x = Py \rightarrow \boldsymbol{y} = \boldsymbol{P^{-1}x}$$
$$x' = PJ(P^{-1}x)$$
$$P^{-1}x' = J(P^{-1}x)$$
$$(P^{-1}x)' = J(P^{-1}x)$$
$$y' = Jy$$

4x4 with one λ

$$y = c_1e^{\lambda t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2e^{\lambda t}\begin{bmatrix}t\\1\\0\\0\end{bmatrix} + c_3e^{\lambda t}\begin{bmatrix}\frac{t^2}{2!}\\t\\1\\0\end{bmatrix} + c_4e^{\lambda t}\begin{bmatrix}\frac{t^3}{3!}\\\frac{t^2}{2!}\\t\\1\end{bmatrix}$$

4x4 with 2 λ's

$$y = c_1e^{\lambda_1 t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2e^{\lambda_1 t}\begin{bmatrix}t\\1\\0\\0\end{bmatrix} + c_3e^{\lambda_2 t}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + c_4e^{\lambda_2 t}\begin{bmatrix}0\\0\\t\\1\end{bmatrix}$$

12.1.1: 2x2 Fundamental Solutions Example

Find the fundamental solutions matrix for the following system.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

12.2: Non-Homogeneous Systems

Non-homogeneous systems refer to systems of differential equations that include a non-zero external input or forcing function, making the system's behavior more complex.

Method of Undetermined Coefficients

- Make "guess" from f
  - x=a*v
- Plug guess in as x into x'=Ax+f
- Solve for guess
- $x=x_c+x_p$

When f **does not** overlap x_c

Variation of Parameters

- Solve for x_c
- Find X (x_c as one matrix)

$$x_p = X\int (X^{-1} \times f)\, dt$$

When f **does** overlap x_c

12.2: Variation of Parameters for 2nd-Order ODEs

- **y" + p(t)y' + q(t)y = F(t)**
- Solve for y_p given $y_c = y_1 + y_2$

$$y_p = y_1 \int \frac{-y_2 \times F(t)}{W} dt + y_2 \int \frac{y_1 \times F(t)}{W} dt$$

12.2.1: 2x2 Non-Homogenous Example

Find the particular solution for the following system.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ 6e^t \end{bmatrix}$$

12.2.2: 4x4 Non-Homogenous Example

Find the general solution of the following system..

$$\vec{x}'(t) = \begin{bmatrix} -2 & 2 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \vec{x}(t).$$

Hint: $\begin{bmatrix} -2 & 2 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}^{-1}$