6 Eigenvalues and eigenvectors; diagonalization

6.1 Linear transformations from V to W versus linear transformation V to V

Find Q, P, N such that $N = Q^{-1}MP$

$$M = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix}$$

$$T(v) = Mv = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}$$

we are solving for $N = Q^{-1}MP$, where P is the basis of where we are coming from & Q is the basis where we are going, like $MP = Q$

$T: \mathbb{R}^4 \to \mathbb{R}^3$ because M is 3×4

1. Find RREF(M)

$$M = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix} \rightsquigarrow \text{RREF}(M) = \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \to \begin{matrix}\text{rank} = 2\\ \text{nullity} = 2\end{matrix}$$

2. Find nullspace of M

$$\begin{matrix}x_1 + 2x_4 = 0\\ x_2 + x_3 + x_4 = 0\end{matrix} \Rightarrow \begin{matrix}x_1 = -2x_4\\ x_2 = -x_3 - x_4\end{matrix} \Rightarrow \text{nullspace: } \left\{\begin{bmatrix}0\\ -1\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-2\\ -1\\ 0\\ 1\end{bmatrix}\right\}$$

3. Find basis for $\mathbb{R}^4$ given M

$$\boxed{P = \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}}$$

from nullspace(M)

Need 2 more linearly independent vectors to span $\mathbb{R}^4$

4. Find basis for $\mathbb{R}^3$ given M

$$w_1 = Mv_1 = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

$$w_2 = Mv_2 = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$$

let w_3 be any linearly independent vector, $w_3 = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$

$$\boxed{Q = \begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}}$$

5. Finding $N = Q^{-1}MP$

$$Q = \begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 0\\ 1 & -1 & 0\end{bmatrix} \quad M = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix} \quad P = \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix}0 & 1/3 & 1/3\\ 0 & 1/3 & -2/3\\ 1 & -2/3 & 1/3\end{bmatrix}$$

$$N = \begin{bmatrix}0 & 1/3 & 1/3\\ 0 & 1/3 & -2/3\\ 1 & -2/3 & 1/3\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 1 & 1 & 5\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} = \boxed{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = N}$$

N is always a matrix the same size as M with 1's along the diagonal with the same rank as M

6.2 Eigenvalues and eigenvectors

Find P, D such that $D = P^{-1}AP$

$$A = \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$$

1. Find $A - \lambda I$

$$A - \lambda I = \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1-\lambda & 0 & 0\\ 0 & 1-\lambda & 0\\ 2 & 0 & 1-\lambda\end{bmatrix}$$

2. Find eigenvalues ($\det(A - \lambda I) = 0$)

$$\det\begin{bmatrix}1-\lambda & 0 & 2\\ 0 & 1-\lambda & 0\\ 2 & 0 & 1-\lambda\end{bmatrix} = (1-\lambda)(\lambda^2 - 2\lambda + 1) + 2(-2)(1-\lambda) = -\lambda^3 + 3\lambda^2 + \lambda - 3$$

$$= (\lambda - 1)(\lambda + 1)(\lambda - 3)$$

$\hookrightarrow \lambda = -1, 1, 3$

3. Find eigenvectors for each eigenvalue

(plug each λ into $A - \lambda I$)

$\lambda = -1$

$$\begin{bmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \quad \begin{matrix}x_1 = -x_3\\ x_2 = 0\end{matrix} \rightsquigarrow v_{\lambda=-1} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix}0 & 0 & 2\\ 0 & 0 & 0\\ 2 & 0 & 0\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \quad \begin{matrix}x_1 = 0\\ x_2 = 1\\ x_3 = 0\end{matrix} \rightsquigarrow v_{\lambda=1} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix}-2 & 0 & 2\\ 0 & -2 & 0\\ 2 & 0 & -2\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0\end{array}\right] \quad \begin{matrix}x_1 = x_3\\ x_2 = 0\end{matrix} \rightsquigarrow v_{\lambda=3} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

4. Form P & D in $D = P^{-1}AP$

D: diagonal matrix with λs on diagonal

P: eigenvectors as columns — make sure the eigenvalue in column 1 of D has the corresponding eigenvector in column 1 in P & etc..

$$\boxed{D = \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}} \quad \boxed{P = \begin{bmatrix}-1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}}$$