5.2 Linear transformations

Now that we have a good understanding of matrices and their properties, let's talk about linear transformations — an abstraction of linear algebra that we use matrices to represent.

Linear transformations are a function between two vector spaces that preserves the operations of vector addition and scalar multiplication. They observe the same closures as vector spaces, but rather than being closed under addition and scalar multiplication within one space, it preserves those properties when **mapping** from one space to another.

Linear transformations simply **transform** vectors, matrices, or polynomials from one space to another, mapping elements in one space to another. We denote linear transformations as $T : V \to W$, where T represents the transformation itself, and then V is the vector spaces we are "coming from" and W is the vector space we are "going to".

Linear transformations are essentially a function where we apply T to every element in V to get something in W. As mentioned before, linear transformations are closed under addition and scalar multiplication, just like vector spaces, which means:

- Let $T : V \to W$, where $p, q \in V$ and $p', q' \in W$
- $T(p + q) = T(p) + T(q) = p' + q'$
- $T(cp) = cT(p) = cp'$

Linear transformations themselves are **mappings**, which are abstract concepts. We are able to **represent** linear transformations with matrices. These matrices themselves are **not** the transformation itself, but rather a representation of the transformation. Let's talk about how to form these matrices based on the transformation.

Forming M_T, the matrix representation of the linear transformation T
When we map from V to W, the dimension of each of those vector spaces tells us what the size of M_T should be, because we must use M_T to map an element in the first space to an element in the second space, which must be well-defined dimension-wise through matrix multiplication.

When we multiply an input element by a matrix representation of a linear transformation, we **always** multiply on the **left** by the matrix of the input:

$$\begin{bmatrix} \\ w \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & M_T & \\ & & \end{bmatrix} \begin{bmatrix} \\ v \\ \\ \end{bmatrix}$$

Further, when we are attempting to find M_T, you are either told:

- what input and output elements are (e.g. $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$)
- the property of the transformation (e.g. $T : P_3 \to P_3$ where $p(x) \mapsto xp'(x)$)

For example, if we are mapping from $V : \mathbb{R}^2$ to $W : P_3$, we are "converting" 2×1 vectors to polynomials up to degree 3. We know what the dimension of each of these spaces are, which will help us form M_T.
$\text{Dim}(\mathbb{R}^2) = 2$
$\text{Dim}(P_3) = 4$
So, in the case of $T : \mathbb{R}^2 \rightarrow P_3$, we will be mapping from a 2-dimension space to a 4-dimensional space. To think about matrix-wise, we need a matrix to left-multiply to transform a 2×1 vector into a 4×1 vector, which would thus make M_T a 4×2 matrix:

$$\begin{bmatrix} w \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} M_T & \\ & \\ & \\ & \\ \end{bmatrix} \begin{bmatrix} v \\ \\ \end{bmatrix}$$

Further, we need to define *how* we are mapping from $\mathbb{R}^2$ to P_3. Let's say that we are told:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x^2 + 3x + 2, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x^3 + 3x^2 + 2$$

Then, we just need to define a matrix, M_T that makes this true. When doing linear transformations with matrices, we represent polynomials as column vectors as well where each entry corresponds to each power. So, we can rewrite the above information as:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

As we can see, the input vectors are the standard basis vectors of $\mathbb{R}^2$. Unless otherwise specified, we want M_T to be with respect to the standard bases of the input and output spaces, so $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $\{1, x, x^2, x^3\}$

Therefore our transformation matrix, $M_T = \begin{bmatrix} 2 & 2 \\ 3 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$ We can see that this computes to be:

$$\begin{bmatrix} 2 & 2 \\ 3 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Where we multiply our inputs in $\mathbb{R}^{\not\models}$ by M_T and get elements in P_3. While this multiplication is obvious, we can use any input vector to right-multiply the transformation matrix and get any output vector.

For example, for this transformation, $T\left(\begin{bmatrix} 5 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 3 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ -7 \\ -4 \end{bmatrix} = 2 + 15x - 7x^2 - 4x^3$

However, let's say we were given the other type of information, where we are told the transformation property and have to compute M_T. This makes a bit more sense when we are mapping from one polynomial space to another.
For example:

$$T : P_3 \to P_2 \text{ such that } T(p) = 2p' - 3p" + p(1)$$

First, we can solve for M_T with respect to the standard basis.
To do this, we will transform each element in the standard input basis, and then represent it in terms of our output basis.
Input standard basis: $\{1, x, x^2, x^3\}$ $(dim = 4)$
Output standard basis: $\{1, x, x^2\}$ $(dim = 3)$

$$T(1) = 2(0) - 3(0) + 1 = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = 2(1) - 3(0) + 1 = 3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x^2) = 2(2x) - 3(2) + 1 = 4x - 5 = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}$$

$$T(x^3) = 2(3x^2) - 3(6x) + 1 = 6x^2 - 18x + 1 = \begin{bmatrix} 1 \\ -18 \\ 6 \end{bmatrix}$$

We are mapping from a dimension 4 space to a dimension 3 space, so our inputs will be 4×1 vectors and the outputs will be 3×1 vectors, which makes our M_T a 3×4 matrix.
We are going to do the exact same thing we did for the previous example to form M_T:

$$M_T = \begin{bmatrix} 1 & 3 & -5 & 1 \\ 0 & 0 & 4 & -18 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -5 & 1 \\ 0 & 0 & 4 & -18 \\ 0 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 & 1 \\ 0 & 0 & 4 & -18 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, we know there are infinitely many bases for P_3 and P_4, so we can write this transformation with respect to different bases.
Input basis: $\{1, x^2, x^3, 3-x\}$
Output basis: $\{1, -5+2x, 1-18x+6x^2\}$

$T(1) = 2(0) - 3(0) + 1 = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x^2) = 2(2x) - 3(2) + 1 = 4x - 5 = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}$

$T(x^3) = 2(3x^2) - 3(6x) + 1 = 6x^2 - 18x + 1 = \begin{bmatrix} 1 \\ -18 \\ 6 \end{bmatrix}$

$T(3-x) = 2(-1) - 3(0) + 2 = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So we know what we want the result of the transformation to be, but we need to represent this output **in terms of the output basis**.
The output basis in vector form is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -18 \\ 6 \end{bmatrix} \right\}$$

So we need to write our transformed vectors in terms of these vectors.

$T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x^3) = \begin{bmatrix} 1 \\ -18 \\ 6 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -18 \\ 6 \end{bmatrix}$

$T(3-x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$ These are each just solving linear systems. You can use row reduction or try to logic it out — whatever works best.
Therefore, our new M_T looks like this:

$$M_T = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

As you can see, linear transformations are a big concept in linear algebra. All matrices perform a linear transformation to some extent. Whether they map to the same space or not is a question for 6.1.

Again, to reiterate, **these matrices are representations of linear transformations, not the transformation itself**. Linear transformations themselves are *mappings*, and we can perform them with matrices.

In MATH 2400, the most common linear transformation questions are with regards to mappings from polynomial spaces to polynomial spaces, $\mathbb{R}$ vector spaces to different spaces, and with $\mathbb{R}^{2\times 2}$.

Linear transformations of $\mathbb{R}^{2\times 2}$

When performing linear transformations with $\mathbb{R}^{2\times 2}$, we must represent these matrices as 4×1 column vectors, not as 2×2 matrices. This is because linear operations, such as a transformation that adds two matrices together, similar to the polynomial examples below, cannot be represented by a 2×2 transformation matrix.

All transformation matrices mapping $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ should be 4×4 matrices.

For example: $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ such that $T(A) = 2A - A^T$

First, start with an **arbitrary** 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Determine $T(A)$: $T(A) = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & 2b-c \\ 2c-b & d \end{bmatrix}$

$$\text{Form } M_T: \quad \begin{matrix} a & b & c & d & \\ \end{matrix}$$
$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Populate M_T: each column represents a, b, c, d, respectively, and each row represents each element in the transformation matrix.

Element 1 ($T(a)_{1,1}$): a

Element 2 ($T(a)_{1,2}$): $2b - c$

Element 3 ($T(a)_{2,1}$): $2c - b$

Element 4 ($T(a)_{2,2}$): d

$$M_T = \begin{matrix} a & b & c & d \end{matrix}$$
$$M_T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

And that's it! This is a very common problem — keep it in mind.

Linear transformations summary

- The size of M_T is output dimension $\times$ input dimension
- Linear transformation matrices, M_T are of the form:

$$\overset{\textbf{coming from}}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}} \textbf{going to}$$

- Linear transformations are mappings closed under addition and scalar multiplication, and also, **linear**
- M_T can be determined either from input and output elements or the definition of the transformation itself
- All transformation matrices mapping $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ should be 4×4 matrices
- Polynomials can be represented as column vectors and treated like $\mathbb{R}^n$ vectors in linear transformations
- Common matrices for $\mathbb{R}^2$ include
  - Rotation matrix: $\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$
  - Reflect across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Reflect across y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Further, the determinant of each of these matrices tells us geometric information about the transformation.
    * <u>Magnitude of determinant</u>: How the area of the shape is scaled, if $Det(A) = 0$, indicates "squashing" to a lower dimension space
    * <u>Sign of the determinant</u>: Negation implies reflection