3 Determinants, vector spaces and subspaces

3.1 Determinants

Find the determinant of the following matrices:

$$\begin{bmatrix} 5 & 9 \\ 6 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 7 & 2 \\ 9 & 5 & 3 \\ 7 & 2 & 6 \end{bmatrix}$$

For a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det = ad - bc$

$$\begin{vmatrix} 5 & 9 \\ 6 & 2 \end{vmatrix} = 5(2) - 9(6) = -44$$

For a 3x3 matrix, we do minor expansion

$$\begin{vmatrix} 4 & 7 & 2 \\ 9 & 5 & 3 \\ 7 & 2 & 6 \end{vmatrix} = 4\begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} - 7\begin{vmatrix} 9 & 3 \\ 7 & 6 \end{vmatrix} + 2\begin{vmatrix} 9 & 5 \\ 7 & 2 \end{vmatrix}$$

$$= 4(5(6) - 3(2)) - 7(9(6) - 3(7)) + 2(9(2) - 5(7))$$

$$= 4(24) - 7(33) + 2(-17)$$

$$= 96 - 231 - 34$$

$$= -169$$

3.2 Vector spaces and subspaces

Consider the vector space $\mathbb{R}^3$. Show that the following subsets either are or are not subspaces of $\mathbb{R}^3$

- The set of vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that $v_1v_3 + v_1v_2 = 0$

Let $p, q \in \mathbb{R}^3$ ($\therefore$ subset of $\mathbb{R}^3$)

$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, $p_1p_2 + p_1p_3 = 0$ $\quad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$, $q_1q_2 + q_1q_3 = 0$

CLOSED UNDER ADDITION

$$p+q \rightarrow (p_1+q_1)(p_2+q_2) + (p_1+q_1)(p_3+q_3) \stackrel{?}{=} 0$$

$$= p_1p_2 + p_1q_2 + p_2q_1 + q_1q_2 + p_1p_3 + p_1q_3 + q_1p_3 + q_1q_3$$

$$= p_1p_2 + p_1p_3 + q_1q_2 + q_1q_3 + p_1q_2 + p_2q_1 + p_1q_3 + q_1p_3$$

$$= 0 + 0 + \underbrace{p_1(q_2+q_3)}_{\text{not necessarily} = 0} + \underbrace{q_1(p_2+p_3)}_{\text{not necessarily} = 0}$$

NOT closed under addition!

We can also ... disprove specifically

Let $p, q \in \mathbb{R}^3$

$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, $p_1p_2 + p_1p_3 = 0 \Rightarrow \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $-1(1) + -1(-1) = 0$

$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ $q_1q_2 + q_1q_3 = 0 \Rightarrow \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$, $0(4) + 0(-3) = 0$

$p+q = \begin{bmatrix} -1+0 \\ 1+4 \\ -1+-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix} \Rightarrow -1(5) + -1(-4) = -1 \neq 0$

$\hookrightarrow$ NOT closed under addition!

- The set of vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that $v_1 - v_2 = 0$

Let $p, q \in \mathbb{R}^3$ ($\therefore$ subset of $\mathbb{R}^3$)

$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, $p_1 - p_2 = 0$ $\quad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$, $q_1 - q_2 = 0$

CLOSED UNDER ADDITION

$$p+q \rightarrow (p_1+q_1) - (p_2+q_2) \stackrel{?}{=} 0$$

$$p_1 - p_2 + q_1 - q_2 \stackrel{?}{=} 0$$

$$0 + 0 = 0 \checkmark$$

CLOSED UNDER SCALAR MULTIPLICATION

$c \in \mathbb{R}$

$cp = \begin{bmatrix} cp_1 \\ cp_2 \\ cp_3 \end{bmatrix}$, $cp_1 - cp_2 \stackrel{?}{=} 0$

$$c(p_1 - p_2) \stackrel{?}{=} 0$$

$$c(0) = 0 \checkmark$$

ZERO VECTOR?

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $0 - 0 = 0 \checkmark$

THIS IS A VECTOR SPACE $\therefore$ A SUBSPACE OF $\mathbb{R}^3$