2 Gaussian elimination, systems of equations and matrix inverses

2.1 Matrices and systems of linear equations

Solve for the RREF of the following augmented system:

$$\left[\begin{array}{ccc|c} 5 & 10 & 5 & 15 \\ 3 & 12 & -9 & 9 \\ 4 & -4 & 12 & 8 \end{array}\right]$$

step 1: get top left 1

$R1 = \frac{1}{5}R1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 12 & -9 & 9 \\ 4 & -4 & 12 & 8 \end{array}\right]$$

step 2: use top left 1 to get zeroes below

$R2 = R2 - 3R1$
$R3 = R3 - 4R1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & -12 & 0 \\ 0 & -12 & 8 & -4 \end{array}\right]$$

step 3: try to make 0 in R3C2

$R3 = R3 + 2R2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & -16 & -4 \end{array}\right]$$

step 4: make 1 in R3C3

$R3 = -\frac{1}{16}R3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

step 5: make 0 in R2C3

$R2 = R2 + 12R3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

step 6: make 1 in R2C2

$R2 = \frac{1}{6}R2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

now we have 2 1's we can use

step 7: use R2 & R3 to get zeroes in R1

$R1 = R1 - R3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 11/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

step 8: Eliminate last non-zero entry

$R1 = R1 - 2R2$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right]}$$

THIS IS OUR RREF!

Since Rank = n, we have a unique solution:

$x_1 = \frac{7}{4}, x_2 = \frac{1}{2}, x_3 = \frac{1}{4}$

Solve the following $Ax = b$ (starting with the RREF) : $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Here, we see we have 5 columns, so our x will be a linear combination of 5x1 vectors, where we can set x_1, x_2, x_3, x_4, x_5

- pivot variables: x_1, x_4, # pivot variables = rank
- free variables: x_2, x_3, x_5. # free variables = nullity

- To find our solution set, we take the following steps:

1: write equations from the RREF

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$1x_1 + 2x_2 + 3x_3 + 0x_4 + 4x_5 = 0$
$1x_4 + 5x_5 = 6$

2: express pivot variables in terms of free variables

$1x_1 + 2x_2 + 3x_3 + 0x_4 + 4x_5 = 0 \Rightarrow x_1 = -2x_2 - 3x_3 - 4x_5$
$1x_4 + 5x_5 = 6 \Rightarrow x_4 = 6 - 5x_5$

3: set up solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}}_{\text{non-homogeneous vector}} + \underbrace{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} x_2 + \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} x_3 + \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} x_5}_{\text{one vector for each free variable}}$$

4: plug in equations into vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 0 \\ 0 \\ -5 \\ 1 \end{bmatrix} x_5$$

$x_1 = -2x_2 - 3x_3 - 4x_5$
$x_4 = 6 - 5x_5$

set each free variable equal to itself

Fill in all equations

plug in zeroes everywhere else

* also written as

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}}$$

2.2 Inverse of a square matrix

Find A^{-1}, if it exists, where $A = \begin{bmatrix} 5 & 10 & 5 \\ 3 & 12 & -9 \\ 4 & -4 & 12 \end{bmatrix}$ (same as RREF example)

Step 1: Augment with identity matrix

$$\left[\begin{array}{ccc|ccc} 5 & 10 & 5 & 1 & 0 & 0 \\ 3 & 12 & -9 & 0 & 1 & 0 \\ 4 & -4 & 12 & 0 & 0 & 1 \end{array}\right]$$

Step 2: Row reduce until A (left matrix) is the identity matrix

$$\left[\begin{array}{ccc|ccc} 5 & 10 & 5 & 1 & 0 & 0 \\ 3 & 12 & -9 & 0 & 1 & 0 \\ 4 & -4 & 12 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 = \frac{1}{5}R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1/5 & 0 & 0 \\ 3 & 12 & -9 & 0 & 1 & 0 \\ 4 & -4 & 12 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 = R3-4R1]{R2 = R2-3R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1/5 & 0 & 0 \\ 0 & 6 & -12 & -3/5 & 1 & 0 \\ 0 & -12 & 8 & -4/5 & 0 & 1 \end{array}\right] \xrightarrow{R3 = R3+2R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1/5 & 0 & 0 \\ 0 & 6 & -12 & -3/5 & 1 & 0 \\ 0 & 0 & -16 & -2 & 2 & 1 \end{array}\right] \xrightarrow{R3 = -\frac{1}{16}R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1/5 & 0 & 0 \\ 0 & 6 & -12 & -3/5 & 1 & 0 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/16 \end{array}\right] \xrightarrow{R2 = R2+12R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1/5 & 0 & 0 \\ 0 & 6 & 0 & 9/10 & -1/2 & -3/4 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/16 \end{array}\right] \xrightarrow{R2 = \frac{1}{6}R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 3/20 & -1/12 & -1/8 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/16 \end{array}\right] \xrightarrow{R1 = R1-R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3/40 & 1/8 & 1/16 \\ 0 & 1 & 0 & 3/20 & -1/12 & 1/8 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/16 \end{array}\right] \xrightarrow{R1 = R1-2R2} \boxed{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/40 & 7/24 & 5/16 \\ 0 & 1 & 0 & 3/20 & -1/12 & -1/8 \\ 0 & 0 & 1 & 1/8 & -1/8 & 1/16 \end{array}\right]}$$

we formed the identity matrix → this is A^{-1}