MATH 2400 Old Finals & Quizzes Questions

1.1: Matrices

1.2: Matrix Arithmetic

2.1: Matrices and Systems of Linear Equations

1. Let A be the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

(a) Find a basis for the row space of A, i.e. for the span of the rows of A.

(b) Find a basis for the image of A, i.e. the column space of A.

(c) Find a basis for $\ker(A)$.

(d) Is the vector $\begin{bmatrix} 0 & 3 & 1 & 2 & 0 \end{bmatrix}$ in the row space of A? Prove or disprove.

Solutions

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_4 - R_5]{R_1 \to R_1 - R_4 - R_5} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{the rows}]{\text{reordering}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_3]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) From the row reduced echelon form, we see that a basis for $\text{row}(A)$ is given by:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix} \right\}.$$

(b) From the row reduced echelon form, we see that a basis for $\text{col}(A)$ is given by:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) From the row reduced echelon form we get that the solution of $A\vec{x} = 0$ is

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_4 \text{ and } x_5 \text{ free variables} \end{cases}$$

and thus the basis of $\ker(A)$ is given by: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) Yes as $\begin{bmatrix} 0 & 3 & 1 & 2 & 0 \end{bmatrix} = 3\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix} \in \text{row}(A)$.

1. Each of the following matrices is the reduced row-echelon form of some system of linear equations. For each of them, find a particular solution of the equation, the [vector space of] solutions to the associated homogenous system (i.e., find the kernel [or nullspace] of the coefficient matrix), and then write the general solution of the linear system.

(a) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2 is a (the) free variable, set it equal to zero and obtain the particular solution $\mathbf{x}_p = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

The homogeneous solutions are $\mathbf{x}_c = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for $t \in \mathbb{R}$.

The general solution is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_c = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for $t \in \mathbb{R}$.

(b) $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Since x_1 is a (the) free variable, set it equal to zero and obtain the particular solution $\mathbf{x}_p = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$.

The homogeneous solutions are $\mathbf{x}_c = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $t \in \mathbb{R}$.

The general solution is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_c = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $t \in \mathbb{R}$.

2.2: The Inverse of a Square Matrix

2. Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

(a) Find a matrix X that satisfies $AX = B$.

Since we can also use it for part (b), observe that $A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$.

Then $X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 1 & 11 \end{bmatrix}$

(b) Find a matrix Y that satisfies $YA = B$.

$Y = BA^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -11 & 9 \end{bmatrix}$

3. Consider the matrix $M(z) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & z \\ 1 & 1 & 3 \end{bmatrix}$.

(a) There is a value of z for which the rank of the matrix $M(z)$ is less than three. What is it?

Looking for z so that $\det(M(z)) = 0$. Expand along 1st row:

$$\det(M(z)) = 2\begin{vmatrix} 2 & z \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & z \\ 1 & 3 \end{vmatrix} = 2(6-z) - (3-z) = -z + 9$$

so the determinant is zero (and the rank is less than 3) for $z = 9$.

(b) For this value of z, find a non-zero vector in the kernel (or nullspace) of $M(z)$.

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 2 & 9 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 2 & 9 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R3 \leftarrow R3 - 2R1]{R2 \leftarrow R2 - R1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & -1 & -6 & 0 \end{array}\right]$$

$$\xrightarrow[R3 \leftarrow R3 + R2]{R1 \leftarrow R1 - R2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the kernel is spanned by $\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$

4. (a) There are two values of λ for which the matrix $Q(\lambda) = \begin{bmatrix} 2-\lambda & 5 \\ 3 & 4-\lambda \end{bmatrix}$ does not have rank 2. What are they?

Looking for λ such that $\det(Q(\lambda)) = 0$. From the 2-by-2 formula we have

$$\det(Q(\lambda)) = (2-\lambda)(4-\lambda) - 15 = \lambda^2 - 6\lambda + 8 - 15 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$$

so the two values of λ are $\lambda = 7$ and $\lambda = -1$.

(b) Call the two values of λ that you found in part (a) λ_1 and λ_2. Calculate the product $Q(\lambda_1)Q(\lambda_2)$.

We'll let $\lambda_1 = 7$ and $\lambda_2 = -1$. Then

$$Q(7)Q(-1) = \begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For the record, if we had let $\lambda_1 = -1$ and $\lambda_2 = 7$ then

$$Q(-1)Q(7) = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

just the same.

3.2: Vector spaces and subspaces

6. (a) Let $\mathcal{P}_4$ be the vector space of polynomials of degree less than or equal to 4. Among the following subsets of $\mathcal{P}_4$, only one is **not** a subspace, which one is it? You must justify your answer. *You do not need to explain why the others are subspaces.*

(i) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(7) = 0\}$ (ii) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(0) = 0\}$
(iii) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p(x) = p(-x)\}$ (iv) $\mathcal{S} = \{p(x) \in \mathcal{P}_4 \mid p'(x) \neq 0\}$

(b) Let $\mathcal{M}_2$ be the vector space 2×2-matrices. Among the following functions $T\colon \mathcal{M}_2 \to \mathbb{R}$, only one does **not** define a linear transformation, which one is it? You must justify your answer. *You do not need to explain why the others are linear.*

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + 3b - c + 5d$ (ii) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)^2 + 1 - (a+b+1)^2$

(iii) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$ (iv) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

(c) Among the following systems, only one is **not** autonomous, which one is it? You must justify your answer. *You do not need to explain why the others are autonomous.*

(i) $\begin{cases} x'(t) = x(t) - 2e^t y(t) \\ y'(t) = x^2(t) + y(t) \end{cases}$ (ii) $\begin{cases} x'(t) = x(t) - y(t) \\ y'(t) = -x(t) + x(t)y(t) \end{cases}$

(iii) $\begin{cases} x'(t) = (\cos^2(t) + 1)x(t) - 2y(t) + \sin^2(t)x(t) \\ y'(t) = -2x(t) + 4y(t) \end{cases}$ (iv) $\begin{cases} x'(t) = y(t) \\ y'(t) = \sin(x(t)) - y(t) \end{cases}$

Solutions

(a) $\boxed{\text{(iv)}}$ The zero vector of $\mathcal{P}_4$ is the zero polynomial $z(x) = 0$. $z'(x) = 0$ so $z(x)$ does not belong to the subset (iv). Since it does not contain the zero vector it cannot be a subspace.

(b) $\boxed{\text{(iv)}}$ Note that the definition of T in *(iv)* is equivalent to the determinant of the matrix. We know that $\det(A+B) \neq \det(A) + \det(B)$ (take the $A, B = I$ if you would like). Therefore the subset is not closed under addition and is not a subspace.

(c) $\boxed{\text{(i)}}$ This system has an explicit dependence on the t variable in the first equation $x(t) - 2e^t y(t)$ so it is not autonomous. Note that (iii) appears to have a dependence on t in the first equation but

$$(\cos^2(t) + 1)x(t) - 2y(t) + \sin^2(t)x(t) = x(t)\cos^2(t) + x(t) - 2y(t) + \sin^2(t)x(t)$$

$$= x(t)(\cos^2(t) + \sin^2(t)) + x(t) - 2y(t) = 2x(t) - 2y(t)$$

so it does not in fact have an explicit t dependence.

1. Let $\mathcal{P}_4$ be the vector space of polynomials of degree at most 4. Two of the following subsets of $\mathcal{P}_4$ are vector subspaces of $\mathcal{P}_4$ and one is not. For the two that are subspaces, calculate the dimension and find a basis of each. For the one that is not a subspace, explain briefly why not. Finally, don't forget to do part (d).

(a) $\mathcal{S} = \{p \in \mathcal{P}_4 \,|\, p \text{ is an even function, i.e., } p(-x) = p(x)\}$

(b) $\mathcal{T} = \{p \in \mathcal{P}_4 \,|\, p(0) + p'(1) = 2\}$

(c) $\mathcal{U} = \{p \in \mathcal{P}_4 \,|\, \text{the coefficients of } p \text{ are "palindromic"}\}$. Here, "palindromic" means that the coefficients read the same forward and backward, so that if $p = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ then $[a_0, a_1, a_2, a_3, a_4] = [a_4, a_3, a_2, a_1, a_0]$.

(d) Find the dimension of and a basis for the intersection of the two subsets that are subspaces.

Write $p \in \mathcal{P}_4$ as $p = p_0 + p_1x + p_2x^2 + p_3x^3 + p^4x^4$. Then
(a) p is even if and only if $p_1 = p_3 = 0$ so $p = p_0 + p_2x^2 + p_4x^4$, so $\dim \mathcal{S} = 3$ and basis is $\{1, x^2, x^4\}$

(b) Not a subspace because the zero polynomial is not in $\mathcal{T}$.

(c) p is palindromic if and only if $p_0 = p_4$ and $p_1 = p_3$, so $p = p_0(1 + x^4) + p_1(x + x^3) + p_2x^2$, so $\dim \mathcal{U} = 3$ and a basis is $\{1 + x^4, x + x^3, x^2\}$

(d) Both conditions in the definitions of $\mathcal{S}$ and $\mathcal{U}$ must hold, so $p_1 = p_3 = 0$ and $p_0 = p_4$, so $p = p_0(1 + x^4) + p_2x^2$ so $\dim \mathcal{S} \cap \mathcal{U} = 2$ and a basis is $\{1 + x^4, x^2\}$.

1. Let $\mathcal{M}_{2\times3}$ be the vector space of 2-by-3 matrices. Two of the following subsets of $\mathcal{M}_{2\times3}$ are vector subspaces of $\mathcal{M}_{2\times3}$ and one is not. For the two that are subspaces, calculate the dimension and find a basis of each. For the one that is not a subspace, explain briefly why not. Finally, don't forget to do part (d).

(a) $\mathcal{S} = \{A \in \mathcal{M}_{2\times3} \,|\, \text{the rows of } A \text{ sum to zero}\}$

Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then the condition says $a + b + c = 0$ and $d + e + f = 0$, so $\mathcal{S}$ is the solution set of a homogeneous system of linear equations, and is thus a vector space. It has dimension 4 and a basis is

$$\left\{ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \right\}$$

(b) $\mathcal{T} = \{A \in \mathcal{M}_{2\times3} \,|\, \text{the columns of } A \text{ sum to zero}\}$

This condition says that $a + d = 0$, $b + e = 0$ and $c + f = 0$. Again, as the solution set to a homogeneous system of linear equations, $\mathcal{T}$ is a vector space. Its dimension is 3 and a basis is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

(c) $\mathcal{U} = \{A \in \mathcal{M}_{2\times3} \,|\, a_{11} = 1\}$

This is not a vector space, since the zero matrix is not in $\mathcal{U}$.

(d) Find the dimension of and a basis for the intersection of the two subsets that are subspaces.

A basis for $\mathcal{S} \cap \mathcal{T}$ is

$$\left\{ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \right\}$$

so the dimension of $\mathcal{S} \cap \mathcal{T}$ is 2.

4.1: Linear Independence, bases, dimension

2. For each following questions, answer True or False. **You need to justify your answer.**

(a) A linear system with fewer unknowns than equations must have either infinitely many solutions or no solutions.

(b) If a vector u is a linear combination of vectors v and w, and v is a linear combination of vectors p, q and r, then u must be a linear combination of p, q, r and w.

(c) For any choice of vectors v_1, v_2 and v_3 in $\mathbb{R}^5$, the subset

$$\{v_1 + 2v_2,\ v_1 - v_2 + 6v_3,\ 4v_2 + v_3,\ v_2 + v_3\}$$

of $\mathbb{R}^5$ can never be linearly independent.

(d) There exists an invertible 10×10 matrix that has 92 entries that are precisely equal to 1.

(e) There exists a 3×3 matrix P such that the linear transformation $T\colon \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ defined by $T(A) = AP - PA$ has rank 9.

Solutions

(a) $\boxed{\text{False}}$ The following system has a unique solution $\begin{cases} x = 2 \\ y = 3 \\ x + y = 5 \end{cases}$

(b) $\boxed{\text{True}}$ If $u = \lambda_1 v + \lambda_2 w$ where $\lambda_1, \lambda_2 \in \mathbb{R}$, and $v = \mu_1 p + \mu_2 q + \mu_3 r$ with $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$, then we get

$$u = \lambda_1\mu_1 p + \lambda_1\mu_2 q + \lambda_1\mu_3 r + \lambda_2 w$$

(c) $\boxed{\text{True}}$ Any 4 vectors in the subspace $\text{Span}(v_1, v_2, v_3) \subseteq \mathbb{R}^5$ must be linearly dependent.

(d) $\boxed{\text{False}}$ There must be at least 2 rows with only 1 entries, and thus by row reduction, we obtain a line of only zero entries, hence the matrix has determinant zero and thus cannot be invertible.

(e) $\boxed{\text{False}}$ We have $T(I_3) = P - P = 0$, thus $I_3 \in \ker(T)$, hence $\text{nullity}(T) \geq 1$. By rank-nullity, we get $\text{rank}(T) \leq 8$.

2. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^6$ spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$.

(a) Show that the set of vectors is linearly independent and hence the set is a basis for $\mathcal{V}$. What is $\dim \mathcal{V}$?

Call the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. In the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the first component is c_1 and the third component is c_2, so if $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, we must have $c_1 = 0$ and $c_2 = 0$, so the vectors are linearly independent. Thus $\dim \mathcal{V} = 2$.

(b) Is there a 6-by-3 matrix L whose image is $\mathcal{V}$? If there is, give an example. If not, explain why not.

The columns of L should span $\mathcal{V}$ and only $\mathcal{V}$, so: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ works.

(c) Is there a 3-by-6 matrix M whose kernel is $\mathcal{V}$? If there is, give an example. If not, explain why not.

Since, $\dim \mathcal{V} = 2$, we would need $\mathrm{rk}(M) = 4$ to satisfy the rank/nullity theorem, but $\mathrm{rk}(M) \leqslant 3$ since it has only 3 columns.

4. Show that the vector $\mathbf{x} = \begin{bmatrix} 9 \\ 5 \\ 3 \\ -7 \\ 7 \end{bmatrix} \in \mathbb{R}^5$ is in the subspace spanned by

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

by expressing $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Solving for c_1, c_2, c_3:

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 9 \\ 1 & 2 & 1 & 5 \\ 1 & 2 & 2 & 3 \\ 4 & 1 & 0 & -7 \\ 1 & 2 & 0 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \\ 0 & -7 & 0 & -35 \\ 0 & -1 & 0 & -5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $\mathbf{x} = -3\mathbf{v}_1 + 5\mathbf{v}_2 - 2\mathbf{v}_3$.

2. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^5$ spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

(a) Show that the set of vectors is linearly independent and hence the set is a basis for $\mathcal{V}$. What is $\dim \mathcal{V}$?

(b) Find a 5-by-3 matrix L whose image is $\mathcal{V}$.

(c) Find a 3-by-5 matrix M whose kernel is $\mathcal{V}$.

(a) The matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ whose rows are the three vectors is in reduced row echelon form, so its rank is 3 and the vectors are linearly independent. Another way to say this is to observe that if the linear combination $c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \\ c_2 \\ c_3 \\ 2c_1 + c_2 + 3c_3 \end{bmatrix}$ is the zero vector, then $c_1 = c_2 = c_3 = 0$ by looking at the first, third and fourth components.

(b) We need the column space of L to be $\mathcal{V}$, so take L to be the transpose of the matrix from the answer to part (a).

(c) The rows of M must be (the transposes of) vectors in the kernel of the matrix from the answer to part (a), which is $\left\{ s \begin{bmatrix} -2 \\ 0 \\ -1 \\ -3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ so for instance $\begin{bmatrix} -2 & 0 & -1 & -3 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. Give a basis for the subspace of the vector space $\mathbb{R}^{2\times 2}$ of 2-by-2 matrices spanned by the set

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

in the vector space $\mathbb{R}^{2\times 2}$ of 2-by-2 matrices. What is the dimension of this subspace?

Call the matrices A, B, C and D. Then it's clear that $D = B + C$ and that A, B and C are independent (could demonstrate this by row-reducing $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$). So $\{A, B, C\}$ is a basis of the subspace and its dimension is 3.

4. Let $M = \begin{bmatrix} 1 & 2 & 4 & 3 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 2 & 0 & 4 & 6 & 6 \end{bmatrix}$.

(a) Find a basis for and the dimension of the rowspace of M.

Subtract 2 times row 1 from row 3 and add 4 times row 2 to row 3 to get $\begin{bmatrix} 1 & 2 & 4 & 3 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so dimension of row space is 2 and a basis for the rowspace is $\{\begin{bmatrix} 1 & 2 & 4 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & -1 \end{bmatrix}\}$.

(b) Find a basis for and the dimension of the nullspace (kernel) of M.

Subtract two times the second row from the first row of the last matrix to get $\begin{bmatrix} 1 & 0 & 2 & 3 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so the dimension of the kernel of M is 3 and a basis for it is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) Find a basis for and the dimension of the column space (range) of M.

We know the dimension of the column space is the same as the dimension of the row space, namely 2, and the first two columns of of M are linearly independent, so we can take the basis of the column space to be

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Also, the leading 1's in the rref of M are in columns 1 and 2.

4.2: Change of Basis, Rank/Nullity Theorem

1. Let $B = \{1, x, x^2\}$ be the standard basis for the vector space $\mathcal{P}_2$ of polynomials of degree at most 2 with real coefficients, and let $C = \{x(x+1)\ , x(x-1)\,,\ (x-1)(x-2)\}$ be another basis for $\mathcal{P}_2$. Find the matrix $P_{C\leftarrow B}$, i.e., the change of basis matrix which satisfies

$$[z]_C = P_{C\leftarrow B}[z]_B$$

for any $z \in \mathcal{P}_2$. As usual, by $[z]_C$ we mean the element z expressed in C coordinates.

To begin, we note that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_B$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_C = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}_B$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_C = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_B$.

This means that

$$P_{B\leftarrow C} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

and $P_{C\leftarrow B} = P_{B\leftarrow C}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -4 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

Thus $P_{C\leftarrow B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$.

(So for instance

$$1 = \frac{1}{2}x(x+1) - x(x-1) + \frac{1}{2}(x-1)(x-2) = \frac{1}{2}x^2 + \frac{1}{2}x - x^2 + x + \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

$$x = \frac{1}{2}x(x+1) - \frac{1}{2}x(x-1) = \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{2}x$$

$$x^2 = \frac{1}{2}x(x+1) + \frac{1}{2}x(x-1) = \frac{1}{2}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x$$

)

3. Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree $\leqslant 3$. Let $\mathcal{W}$ be the subspace of $\mathcal{P}_3$ consisting of polynomials that satisfy $p(1) = 0$ and $p(-1) = 0$. One basis of this subspace is $\mathcal{B}_1 = \{1 - x^2, x - x^3\}$. Another basis is $\mathcal{B}_2 = \{1 - x - x^2 + x^3, 2 + x - 2x^2 - x^3\}$

(a) What polynomial is denoted $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}_1}$?

The polynomial is $2(1 - x^2) + 3(x - x^3) = 1 + 3x - 2x^2 - 3x^3$.

(b) What polynomial is denoted $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}_2}$?

The polynomial is $2(1 - x - x^2 + x^3) + 3(2 + x - 2x^2 - x^3) = 8 + x - 8x^2 - x^3$.

(c) Express the polynomials in the basis $\mathcal{B}_2$ in terms of the basis $\mathcal{B}_1$

It's easy to see that $1 - x - x^2 + x^3 = 1(1 - x^2) - (x - x^3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{B}_1}$ and that

$2 + x - 2x^2 - x^3 = 2(1 - x^2) + 1(x - x^3) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}_1}$

(d) What is the change of basis matrix $P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1}$?

From part (c), $P_{\mathcal{B}_1 \leftarrow \mathcal{B}_2} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. Therefore $P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1} = P^{-1}_{\mathcal{B}_1 \leftarrow \mathcal{B}_2} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

5.1: Consequences of the Rank/Nullity Theorem

5.2: Linear Transformations

5. Let $\mathcal{M}_{2\times 2}$ be the vector space of 2×2-matrices. Let $T\colon \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be the linear transformation defined as follows:

$$T(A) = \begin{bmatrix} \mathsf{tr}(A) & 0 \\ 0 & \mathsf{tr}(A) \end{bmatrix}$$

where $\mathsf{tr}(A)$ is the trace of a matrix A.

(a) Compute $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$.

(b) Find the matrix of T with respect to the standard basis of $\mathcal{M}_{2\times 2}$.

(c) Find a basis for the kernel of T (be sure to express the basis in terms of elements in $\mathcal{M}_{2\times 2}$ and not as column vectors).

(d) What are the eigenvalues of T?

(e) Is T diagonalizable? If yes, find a basis in $\mathcal{M}_{2\times 2}$ that diagonalizes T (be sure to express the basis in terms of elements in $\mathcal{M}_{2\times 2}$ and not as column vectors). If not, can you find a Jordan canonical form?

Solutions

(a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) We evaluate T on the standard basis

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we obtain:

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

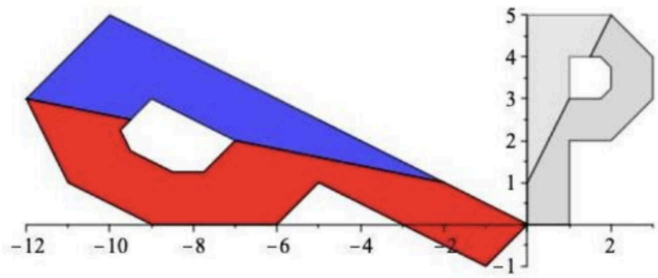
(c) We can either row reduce $[T]$, or alternatively, from (b), we already know that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are in $\ker(T)$ and it's easy to guess that $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is also in the $\ker(T)$. As the image of T is non-trivial, we know the nullity of T is at most 3, and thus actually equals 3, and we obtain that the basis of $\ker(T)$ is $\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

(d) From question (a), we know $\lambda = 2$ is an eigenvalue with eigenvector $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. From question (c), we know $\lambda = 0$ is an eigenvalue with multiplicity of at least 3 (with eigenvectors the basis of $\ker(T)$). Since $[T] \in \mathcal{M}_{4\times 4}$, there can not be more eigenvalues.

(e) Yes T is diagonalizable as the algebraic multiplicity of each eigenvalue equals the geometric multiplicity (the dimension of $\ker([T] - \lambda I_4)$). The diagonalizing basis are the vectors (a) and (c) which are:

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

2. (a) Let A be the matrix that maps the point $(x, y) \in \mathbb{R}^2$ to the point $L(x, y) \in \mathbb{R}^2$ via the map $L(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix}$. If L maps the upright gray **P** to the red and blue one in the following figure, what is the matrix A?



From the picture, $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$. Therefore

$$A = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

(b) Calculate $\det(A)$ and interpret it geometrically in terms of the figure above.

$\det(a) = -3$, therefore A (and hence L) multiplies areas by 3 and flips the orientation.

3. Let $T\colon \mathcal{P}_3 \to \mathcal{P}_2$ be the mapping defined by $T(p) = p' + p'' + p(1)$.

(a) Find the matrix of T with respect to the standard bases of $\mathcal{P}_3$ and $\mathcal{P}_2$.

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathcal{P}_3$. Then

$$\begin{aligned} T(p) &= (a_1 + 2a_2x + 3a_3x^2) + (2a_2 + 6a_3x) + (a_0 + a_1 + a_2 + a_3) \\ &= (a_0 + 2a_1 + 3a_2 + a_3) + (2a_2 + 6a_3)x + 3a_3x^2 \end{aligned}$$

so the matrix of T with respect to the standard basis is

$$M_T = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) Find the matrix of T with respect to the basis $\{1, x^2, x^3, 2 - x\}$ of $\mathcal{P}_3$ and the basis $\{1, 3 + 2x, 1 + 6x + 3x^2\}$ of $\mathcal{P}_2$.

Since $T(1) = 1$, $T(x^2) = 3 + 2x$, $T(x^3) = 1 + 6x + 3x^2$ and $T(2 - x) = 0$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

6.1: Linear Transformations: T: V → W and T: V → V

1. Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that

$$L\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\quad L\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\quad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)$, $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$

Since $\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\1\\1\end{bmatrix}$, $L\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$.

Since $\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\\0\end{bmatrix}$.

Since $\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^3$ and $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

The columns of M_L are the images of the (standard) basis vectors and we have two of them. We still need to observe that since $\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix}$.

Therefore

$$M_L=\begin{bmatrix}-1&0&1\\-1&0&1\\0&-1&1\\0&-1&1\end{bmatrix}$$

(c) What is the rank of L (which is the same as the rank of M_L)? How do you know?

The image of L contains of all vectors in $\mathbb{R}^4$ for which $x_1=x_2$ and $x_3=x_4$, which is 2-dimensional. (So the rank of L is 2.

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

By the rank-nullity theorem the dimension of the kernel is $3-2=1$. In fact, it is spanned by the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ since all the rows of M_L sum to zero.

1. Let L be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}\quad L\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}\quad L\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}\quad L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

(a) Calculate $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$, and $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)$

Since $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Since $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Since $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}-\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$.

(b) Find the matrix M_L of L with respect to the (standard) bases $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^4$ and $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$

The columns of M_L are the images of the (standard) basis vectors and we have three of them. We still need to observe that since $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Therefore

$$M_L=\begin{bmatrix}1&0&0&0\\-1&1&1&0\end{bmatrix}$$

(c) What is the rank of L (which is the same as the rank of M_L)? How do you know?

The image of L contains a basis of $\mathbb{R}^2$ (the middle two given facts are the standard basis of $\mathbb{R}^2$) so the rank of L is 2.

(d) What is the dimension of the kernel (nullspace) of L? How do you know?

By the rank-nullity theorem the dimension of the kernel is $4-2=2$.

1. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ that satisfies:

$$T\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\2\\1\\0\end{bmatrix}\quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

(a) Find the matrix of T with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.

The matrix M of T should satisfy $MP=Q$, where $P=\begin{bmatrix}2&1&1\\1&0&1\\0&1&0\end{bmatrix}$ and $Q=\begin{bmatrix}1&0&1\\-1&2&1\\0&1&1\\1&0&1\end{bmatrix}$.

We need P^{-1}:

$$\left[\begin{array}{ccc|ccc}2&1&1&1&0&0\\1&0&1&0&1&0\\0&1&0&0&0&1\end{array}\right]\xrightarrow{R1\to R2,R2\to R3,R3\to R1}\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&0&0&0&1\\2&1&1&1&0&0\end{array}\right]$$

$$\xrightarrow{R3\to R3-2R1-R2}\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&0&0&0&1\\0&0&-1&1&-2&-1\end{array}\right]\xrightarrow[R3\to-R3]{R1\to R1+R3}\left[\begin{array}{ccc|ccc}1&0&0&1&-1&-1\\0&1&0&0&0&1\\0&0&1&-1&2&1\end{array}\right]$$

Then

$$M=QP^{-1}=\begin{bmatrix}1&0&1\\-1&2&1\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}1&-1&-1\\0&0&1\\-1&2&1\end{bmatrix}=\begin{bmatrix}0&1&0\\-2&3&4\\-1&2&2\\0&1&0\end{bmatrix}$$

(b) Calculate the rank and nullity of T and show that the rank/nullity theorem is satisfied.

Rank and nullity of T are the same as those of M:

$$\begin{bmatrix}0&1&0\\-2&3&4\\-1&2&2\\0&1&0\end{bmatrix}\xrightarrow[R4\to R4-R2]{R1\to R2\to R3}\begin{bmatrix}-1&2&2\\0&1&0\\-2&3&4\\0&0&0\end{bmatrix}\xrightarrow[R1\to-R1]{R3\to R3-2R1-R2}\begin{bmatrix}1&-2&-2\\0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

Rank is 2, Nullity is 1, $2+1=3$.

4. Let $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ be the linear transformation that takes the polynomial $p \in \mathcal{P}_3$ to the polynomial

$$L(p) = p + p' - 2p''$$

(a) Find the matrix of L with respect to the standard basis of $\mathcal{P}_3$.

Calculate:

$$L(1) = 1 \qquad L(x) = 1 + x \qquad L(x^2) = -4 + 2x + x^2 \qquad L(x^3) = -12x + 3x^2 + x^3$$

Therefore, if $p = a_0 + a_1x + a_2x^2 + a_3x^3$ then

$$L(p) = (a_0 + a_1 - 4a_2) + (a_1 + 2a_2 - 12a_3)x + (a_2 + 3a_3)x^2 + a_3x^3$$

Therefore

$$M_L = \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 1 & 2 & -12 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find the Jordan canonical form of this matrix (you only need to find J, not the change of basis matrix P).

Since M is upper triangular, its eigenvalues are its diagonal entries, namely four 1's. It's easy to see that the rank of $M - \mathbf{I}$ is 3, so there is only one Jordan block corresponding to $\lambda = 1$:

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.2: Eigenvalues and Eigenvectors

2. Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(a) What are the rank and nullity of M?

Since the last two rows of M are multiples of the first row, the rank of M is 1. Therefore the nullity is $3 - 2 = 1$.

(b) What are the eigenvalues of M?

$\lambda = 0$ is an eigenvalue of M with multiplicity 2. The other eigenvalue of M is $\text{tr}(M) = 6$ with multiplicity 1.

(c) Is M diagonalizable? Explain how you know.

M is diagonalizable because $\mathbb{R}^3$ has a basis of eigenvectors of M – the two vectors that span the nullspace of M plus the eigenvector for $\lambda = 6$ (which happens to be $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$).

2. Let $M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$

(a) What are the rank and nullity of M?

Since every row of M is a multiple of the first (nonzero) row, the rank of M is 1. Therefore the nullity is $6 - 1 = 5$.

(b) What are the eigenvalues of M?

$\lambda = 0$ is an eigenvalue of M with multiplicity 5. The other eigenvalue of M is tr$(M) = 12$.

(c) Is M diagonalizable? Explain how you know.

M is diagonalizable because $\mathbb{R}^6$ has a basis of eigenvectors of M – the five vectors that span the nullspace of M plus an eigenvector for $\lambda = 12$.

4. Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$ (you don't have to find P^{-1} unless you want to use it to check your work).

$\chi(\lambda) = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$ so the eigenvalues are $\lambda = 4$ and $\lambda = -1$.

For $\lambda = 4$:

$$M - 4\mathbf{I} = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$$

so an eigenvector for $\lambda = 4$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

For $\lambda = -1$:

$$M + \mathbf{I} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

so an eigenvector for $\lambda = -1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Therefore

$$P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

Check:

$$PDP^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 3 & -2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 10 \\ 15 & 10 \end{bmatrix} = M$$

7.1: Diagonalization and what can go wrong

3. Let $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

The eigenvalues of A are $\lambda = 4$ and $\lambda = -1$ (with multiplicity 3).

For $\lambda = 4$, we have $A - 4\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 1 & -1 & 1 & -5 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, we have $A + \mathbf{I} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$, which has rank 2, so there are only two linearly independent eigenvectors, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

We have the eigenvectors for $\lambda = 4$ and $\lambda = -1$, and we need a generalized eigenvector for $\lambda = -1$.

Since $(A + \mathbf{I})^2 = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}_2 = (A + \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and so $P = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

2. Let A be a 4×4 matrix of integers. Assume that A has two eigenvalues; one λ_1 with algebraic multiplicity 1 and another λ_2 with algebraic multiplicity 3 (so the characteristic polynomial of A factors as $(\lambda - \lambda_1)(\lambda - \lambda_2)^3$). Say we know that the dimension $\dim(\ker(A - \lambda_1 I)) = 1$ and $\dim(\ker(A - \lambda_2 I)) = 1$.

(a) Find the Jordan canonical form J for A in terms of λ_1 and λ_2.

Because $\dim(\ker(A - \lambda_2 I)) = 1$, there is only one Jordan block with eigenvalue λ_2. Therefore

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

(b) With A as above, assume further that the determinant and trace of A are $\det(A) = 8$, $\text{tr}(A) = 7$. Find λ_1 and λ_2.

We need $\lambda_1 + 3\lambda_2 = 7$ and $\lambda_1\lambda_2^3 = 8$. It would seem that $\lambda_1 = 1$ and $\lambda_2 = 2$.

3. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

(a) Find the Jordan canonical form J of A.

Since A is lower triangular, the characteristic polynomial is $(3-\lambda)(1-\lambda)^2$ so the eigenvalues are $\lambda = 3$ and $\lambda = 1$. And since $A - \mathbf{I} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ has rank 2 (and nullity 1), the Jordan form of A is

$$J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

Since $A - 3\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{bmatrix}$, the eigenvector for $\lambda = 3$ is $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.
From $A - \mathbf{I}$ computed in part (a) we see that the only linearly independent eigenvector for $\lambda = 2$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then, $(A - \mathbf{I})^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$, which of course has rank 1, and its kernel is the span of $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. So we can take $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as the first generalized eigenvector in the chain and then $(A - \mathbf{I}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So the matrix $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

$$e^{tJ} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

1. Let $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

The characteristic polynomial of A is

$$\chi(\lambda) = \det \begin{bmatrix} -\lambda & 2 & 0 \\ 0 & 1-\lambda & 0 \\ -1 & 2 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda)^2$$

so the eigenvalues of A are $\lambda = 0$ (with multiplicity 1) and $\lambda = 1$ (with multiplicity 2).
For $\lambda = 0$,

$$A - 0\mathbf{I} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to $\lambda = 0$ is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 1$,

$$A - \mathbf{I} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so two linearly independent eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, we can take

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

2. Let A be a 2-by-2 matrix that satisfies the equation $A^2 = A$.

(a) Using a string of three or four equations that starts "$J^2 =$" and ends "$= J$", explain why the Jordan canonical form of A satisfies $J^2 = J$.

(b) What is (are) the only possible eigenvalue(s) of A?

(c) What are the possible Jordan canonical forms of A? (A diagonal matrix is a Jordan matrix with 1-by-1 blocks).

(d) If the rank of A is 2, what is (are) the only possible value(s) of A?

(a) $J^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P = P^{-1}AP = J$.

(b) Suppose $A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$. Then $A^2\mathbf{v} = \lambda^2\mathbf{v} = A\mathbf{v} = \lambda\mathbf{v}$, so we must have $\lambda^2 = \lambda$, and the only solutions of this equation are $\lambda = 0$ and $\lambda = 1$.

(c) If $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $J^2 \neq J$. Therefore A must be diagonalizable and the only possible diagonal matrices with $D^2 = D$ are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) The only rank 2 diagonal form is the identity matrix $\mathbf{I}$, in which case $A = P\mathbf{I}P^{-1} = PP^{-1} = \mathbf{I}$, so A must be the identity matrix.

3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

The eigenvalues of A are $\lambda = 2$ and $\lambda = 1$ (with multiplicity 2).
For $\lambda = 2$, we have $A - 2\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.
For $\lambda = 1$, we have $A - \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, which has rank 2, so there is only one linearly independent eigenvector, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is $J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

We have the eigenvectors for $\lambda = 2$ and $\lambda = 1$, and we need a generalized eigenvector for $\lambda = 1$. Since $(A - \mathbf{I})^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_2 = (A - \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and so $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

The solution of $\mathbf{y}' = J\mathbf{y}$ is $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$. Or, you could say that $e^{tJ} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

The solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = P\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ 1 \\ -t \end{bmatrix}$.

3. A 5-by-5 matrix A has the following properties:

- The characteristic polynomial of A is $(\lambda-6)^3(\lambda+2)^2$
- The nullity of $A-6\mathbf{I}$ is 2
- The nullity of $A+2\mathbf{I}$ is 2

What is the Jordan canonical form of A?

Let J be the Jordan canonical form of A. There are three 6's on the diagonal of J and two -2's.

Since the nullity of $A-6\mathbf{I}$ is 2, there are two Jordan blocks with 6's on their diagonals. Therefore the $\lambda=3$ part of J consists of a two 2-by-2 Jordan block and a 1-by-1 block.

Since the nullity of $A+2\mathbf{I}$ is 2, there are two linearly independent eigenvectors corresponding to $\lambda=-2$, and so the $\lambda=-2$ part of J is diagonal:

$$J=\begin{bmatrix} 6 & 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

4. Recall that $\mathcal{P}_3$ is the (four-dimensional) vector space of polynomials of degree at most 3.

(a) Let $p(x)=a_0+a_1x+a_2x^2+a_3x^3$ be an element of $\mathcal{P}_3$. Let $R\colon \mathcal{P}_3\to\mathcal{P}_3$ be the mapping that takes $p(x)$ to $x^3p\left(\frac{1}{x}\right)$. Show that R is linear mapping from $\mathcal{P}_3$ to $\mathcal{P}_3$.

(b) Let $L\colon \mathcal{P}_3\to\mathcal{P}_3$ be the linear transformation that takes the polynomial $p(x)$ to the polynomial $2R(p)-p$, and let M be the matrix of L with respect to the standard basis of $\mathcal{P}_3$.

Calculate the Jordan canonical form of M – that is, find a Jordan form matrix J so that $M=PJP^{-1}$ for some invertible matrix P. You don't have to find P or P^{-1}, just J.

(a) We have $R(p)(x)=x^3\left(a_0+a_1\frac{1}{x}+a_2\frac{1}{x^2}+a_3\frac{1}{x^3}\right)=a_3+a_2x+a_1x^2+a_0x^3$, so R reverses the coefficients of $p(x)$. This is clearly additive and respects scalar multiplication, in fact the matrix of R (with respect to the standard basis) is $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(b) We have $L(1)=-1+2x^3$, $L(x)=-x+2x^2$, $L(x^2)=2x-x^2$, and $L(x^3)=2-x^3$, so $M=\begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix}$. Since M is symmetric, it is diagonalizable. The characteristic polynomial of M is $(\lambda-1)^2(\lambda+3)^2$, so the eigenvalues of M are $\lambda=1$ and $\lambda=-3$, both with multiplicity 2. So the Jordan form of M is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

4. Let $\mathcal{P}_3$ be the vector space of polynomials of degree 3 or less, and consider the inner product on $\mathcal{P}_3$ defined by

$$\langle p\,,\,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

for $p, q \in \mathcal{P}_3$.

Let $\mathcal{S}$ be the subspace of $\mathcal{P}_3$ spanned by $\{1, x\}$. Express the polynomial $z(x) = 20x^3+12x^2+18x$ as the sum of two polynomials $p_1(x) + p_2(x)$, where $p_1 \in \mathcal{S}$ and p_2 is perpendicular to $\mathcal{S}$.

First we calculate $\langle 1\,,\,x\rangle = \int_{-1}^{1} x\,dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0$, so that $\{1.x\}$ is an orthogonal basis for $\mathcal{S}$. The polynomial p_1 should be the projection of z onto $\mathcal{S}$:

$$\begin{aligned} p_1 = \operatorname{proj}_{\mathcal{S}} z &= \left(\frac{\langle 1\,,\,z\rangle}{\langle 1\,,\,1\rangle}\right)1 + \left(\frac{\langle x\,,\,z\rangle}{\langle x\,,\,x\rangle}\right)x \\ &= \left(\frac{\int_{-1}^{1} 20x^3+12x^2+18x\,dx}{\int_{-1}^{1} 1^2\,dx}\right)1 + \left(\frac{\int_{-1}^{1} 20x^4+12x^3+18x^2\,dx}{\int_{-1}^{1} x^2\,dx}\right)x \\ &= \left(\frac{(5x^4+4x^3+9x^2)\big|_{-1}^{1}}{x\big|_{-1}^{1}}\right)1 + \left(\frac{(4x^5+3x^4+6x^3)\big|_{-1}^{1}}{\frac{1}{3}x^3\big|_{-1}^{1}}\right)x \\ &= \frac{8}{2}1 + \frac{20}{\frac{2}{3}}x = 4 + 30x \end{aligned}$$

And then

$$p_2 = z - p_1 = (20x^3+12x^2+18x) - (30x+4) = 20x^3+12x^2-12x-4$$

Of course $z = p_1 + p_2$, and we can check that

$$\langle p_2\,,\,1\rangle = \int_{-1}^{1} 20x^3+12x^2-12x-4\,dx = 0+8-0-8 = 0$$

and

$$\langle p_2\,,\,x\rangle = \int_{-1}^{1} 20x^4+12x^3-12x^2-4x\,dx = 8+0-8-0 = 0$$

so that p_2 really is orthogonal to $\mathcal{S}$.

4. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, and let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{x}$, with $\mathbf{v} \in \mathcal{S}$ and $\mathbf{w}$ perpendicular to $\mathcal{S}$ (with respect to the standard inner product on $\mathbb{R}^4$).

It's not hard to see that an orthonormal basis for $\mathcal{S}$ is $\left\{ \mathrm{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathrm{e}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Then

$$\mathbf{v} = \mathrm{proj}_{\mathcal{S}} \mathbf{x} = \langle \mathbf{x}, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathbf{x}, \mathrm{e}_2 \rangle \mathrm{e}_2 = \frac{1}{3} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

And then

$$\mathbf{w} = \mathbf{x} - \mathbf{v} = \frac{1}{3} \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

7. The Texas state legislature decides there are too many medical procedures going on in their state that they don't approve of. They decide to impose some new conditions on which procedures can be done based on the following three variables:

- The time t in weeks that a person has needed the procedure.
- The net wealth w in hundreds of thousands of dollars of the person seeking the procedure.
- The contributions d in thousands of dollars the person has made to members of the legislature in the past year.

The legislature passes a law saying that the vector $v = (t, w, d)$ corresponding to the person requesting the procedure must be at distance at most 10 from the subspace W of $\mathbb{R}^3$ spanned by the vectors $w_1 = (0, 1, 1)$ and $w_2 = (0.1, 1, 0)$. If $v = (10, 8, 10)$, will the procedure be allowed?

Ignoring all the verbiage, the problem is asking whether the (perpendicular) distance from the point v to the subspace W spanned by w_1 and w_2 is greater than or less than 10. This distance is the length of the vector $v - \mathrm{proj}_W v$. To calculate the projection, we first need an orthogonal basis for W, so we'll replace w_2 with

$$\widehat{w_2} = w_2 - \frac{\langle w_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = (0.1, 1, 0) - \frac{1}{2}(0, 1, 1) = (0.1, 0.5, -0.5)$$

Then

$$v - \mathrm{proj}_W v = v - \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v, \widehat{w_2} \rangle}{\langle \widehat{w_2}, \widehat{w_2} \rangle} \widehat{w_2}$$
$$= (10, 8, 10) - \frac{18}{2}(0, 1, 1) - (0, 0, 0) = (10, -1, 1)$$

This vector clearly has length bigger than 10 (it's $\sqrt{102}$), so the procedure will **not** be allowed.

2. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Express the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 4 \end{bmatrix}$ as a sum of two vectors $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in \mathcal{S}$ and $\mathbf{y}$ is perpendicular to $\mathcal{S}$.

(b) Find a basis for the orthogonal complement $\mathcal{S}^{\perp}$ of $\mathcal{S}$.

(a) We need to project $\mathbf{v}$ onto $\mathcal{S}$, and for this we need an orthogonal basis of $\mathcal{S}$. Let $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and replace $\mathbf{w}_2$ with $\mathbf{w}_2 - \dfrac{\langle \mathbf{w}_1 , \mathbf{w}_2 \rangle}{\langle \mathbf{w}_1 , \mathbf{w}_1 \rangle} \mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Call this last vector $\mathbf{w}_3$. Then

$$\mathbf{x} = \text{proj}_{\mathcal{S}} \mathbf{v} = \frac{\langle \mathbf{v} , \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1 , \mathbf{w}_1 \rangle} \mathbf{w}_1 + \frac{\langle \mathbf{v} , \mathbf{w}_3 \rangle}{\langle \mathbf{w}_3 , \mathbf{w}_3 \rangle} \mathbf{w}_3 = \frac{8}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{6}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix}$$

and

$$\mathbf{y} = \mathbf{v} - \text{proj}_{\mathcal{S}} \mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

(b) $\mathcal{S}^{\perp}$ is the kernel of the matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, which has reduced row-echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ so the basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ (note that this is an orthogonal, but not orthonormal basis).

8.1: Inner products & Linear transformations, Self-Adjoint and Symmetric matrices

7. (a) Determine the 4×4 matrix P which orthogonally projects a vector in $\mathbb{R}^4$ onto the subspace spanned by $\vec{w} = \begin{bmatrix} -2 \\ 0 \\ 6 \\ -3 \end{bmatrix}$.

(b) Determine the rank of P and the dimension of the null space (i.e. kernel) of P.

(c) Find $P^8 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Solutions

(a) We can view this as projection onto a one dimensional subspace. Recall that the projection matrix onto $\vec{w}$ is given by $P = \frac{1}{\vec{w}^T\vec{w}}\vec{w}\vec{w}^T$. Therefore we get:

$$P = \frac{1}{49}\begin{bmatrix} -2 \\ 0 \\ 6 \\ -3 \end{bmatrix}\begin{bmatrix} -2 & 0 & 6 & -3 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 4 & 0 & -12 & 6 \\ 0 & 0 & 0 & 0 \\ -12 & 0 & 36 & -18 \\ 6 & 0 & -18 & 9 \end{bmatrix}$$

(b) P maps from the 4 dimensional space $\mathbb{R}^4$ onto the one dimensional space *span* $\vec{w}$. Therefore the kernel must be dimension 3 and by rank-nullity the rank is $4-3=1$.

(c) Since P is a projection matrix, $P^2 = P$. As a result $P^8 = P$ and

$$P^8\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = P\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 10 \\ 0 \\ -30 \\ 15 \end{bmatrix}$$

3. There is a 2-by-2 matrix S with the following properties:

- S is symmetric
- $S\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$
- $\text{tr}(S) = 3$

What is S?

We see that $\lambda = 4$ is an eigenvalue of S. Because $\text{tr}(S) = 3$, the other eigenvalue is $\lambda = -1$. And because S is symmetric, the eigenvector $\mathbf{w}$ for $\lambda = -1$ is orthogonal to the eigenvector $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\lambda = 3$. So we can let $\mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Then, using the spectral decomposition we have

$$A = \frac{4}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}\mathbf{v}^T + \frac{-1}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}\mathbf{w}^T = \frac{4}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} + \frac{-1}{5}\begin{bmatrix} 15 & 10 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

1. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

The characteristic polynomial of A turns out to be $\chi_A = -\lambda(\lambda - 3)^2$.

(a) How do you know that A is diagonalizable just by looking at it?

(b) Find a diagonal matrix D and an *orthogonal* matrix P such that $A = PDP^T$.

(a) Because A is symmetric, we know it is diagonalizable.

(b) From χ_A, we see that the eigenvalues of A are 0 (with multiplicity 1) and 3 (with multiplicity 2). Therefore $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

For $\lambda = 0$, since all the rows of A sum to 0, we conclude that the kernel of A is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 3$, the entries of $A - 3\mathbf{I}$ are all -1, so the kernel of $A - 3\mathbf{I}$ is spanned by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Now $\mathbf{v}_2$ and $\mathbf{v}_3$ are both orthogonal to $\mathbf{v}_1$, but not to each other. So we replace $\mathbf{v}_3$ by $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_2 , \mathbf{v}_3 \rangle}{\langle \mathbf{v}_2 , \mathbf{v}_2 \rangle} \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$. Or, just to simplify arithmetic, we can use $2\mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ instead of $\mathbf{w}_3$. Finally, since $\|\mathbf{v}_1\| = \sqrt{3}$, $\|\mathbf{v}_2\| = \sqrt{2}$ and $\|2\mathbf{w}_3\| = \sqrt{6}$, we get

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

8.2: SVD and PCA

1. Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$. What are the singular values of A? In the singular value decomposition $A = Q\Sigma P^T$, what is the matrix Σ?

First we need $A^T A = \begin{bmatrix} 30 & 12 \\ 12 & 30 \end{bmatrix}$. This 2-by-2 matrix has eigenvalues 42 and 18, since column sums are both 42 and trace is 60. Therefore the singular values are $\sigma_1 = \sqrt{42}$ and $\sigma_2 = \sqrt{18} = 3\sqrt{2}$, and

$$\Sigma = \begin{bmatrix} \sqrt{42} & 0 \\ 0 & 3\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9.1: First Order Differential Equations & Wronskian

1. Does there exist non-zero constants C_1, C_2, C_3 such that

$$C_1 x^2 = C_2 \sin x + C_3 \cos x$$

on some interval I? Justify your answer. [*Hint* What does the Wronskian tell you?]

This is asking if $x^2, \sin x, \cos x$ are linearly independent. Calculate the Wronskain.

$$W(x^2, \sin x, \cos x) = \begin{vmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x \end{vmatrix} = x^2(-\cos^2 x - \sin^2 x) - 2x(-\sin x \cos x + \sin x \cos x) + 2(-\mathrm{s}$$

The Wronskian is non-zero so the functions are linearly independent and no such C_i can exist.

8. Do there exist constants $c_1, c_2, c_3 \in \mathbb{R}$, where at least one of them is non-zero, such that:

$$c_1 \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \\ \cos t \end{bmatrix} = c_3 \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}$$

on some interval of (a, b) where $a < b$? Justify your answer.

Solutions Let

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} \cos t \\ \sin t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_3(t) = \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}.$$

We compute their Wronskian:

$$W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(t) = \begin{vmatrix} \sin t & \cos t & \cos t \\ \cos t & \sin t & 0 \\ \cos t & \cos t & 0 \end{vmatrix} = \cos t \begin{vmatrix} \cos t & \sin t \\ \cos t & \cos t \end{vmatrix} = \cos^2 t(\cos t - \sin t).$$

As $W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(t)$ is not constantly zero (for instance at $t = 0$), the vector functions $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent. Thus the constants c_1, c_2, c_3 must be all equal to zero.

2. (a) Find the general solution of $y'' + 4y' + 3y = 0$.

The auxiliary equation is $r^2+4r+3 = (r+3)(r+1) = 0$, so the general solution is $y = c_1e^{-3t}+c_2e^{-t}$.

(b) Find the general solution of $y'' + 4y' + 3y = 6e^{-t}$.

Since e^{-t} is a solution of the homogeneous equation, we guess $y_p = Ate^{-t}$. Then $y_p' = -Ate^{-t} + Ae^{-t}$ and $y_p'' = Ate^{-t} - 2Ae^{-t}$ and so

$$y_p'' + 4y_p' + 3y_p = 2Ae^{-t}$$

which is supposed to be $6e^{-t}$, so $A = 3$ The general solution is

$$y = c_1e^{-3t} + c_2e^{-t} + 3te^{-t}.$$

(c) Solve the initial-value problem $y'' + 4y' + 3y = 6e^{-t}$, $y(0) = 5$, $y'(0) = 0$.

From the general solution in part (b), we have $y(0) = c_1 + c_2$ and $y'(0) = -3c_1 - c_2 + 3$. So we need to solve

$$\begin{aligned} c_1 + c_2 &= 5 \\ -3c_1 - c_2 &= -3 \end{aligned}$$

and we get $c_1 = -1$ and $c_2 = 6$. The (unique) solution of the initial-value problem is

$$y = 6e^{-t} + 3te^{-t} - e^{-3t}$$

2. (a) Find the general solution of $y'' - 2y' + 10y = 0$.

The auxiliary equation is $r^2 - 2r + 10 = 0$, which has complex roots: $\lambda = \dfrac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$ so the general solution is $y = c_1e^t\cos 3t + c_2e^t \sin 3t$.

(b) Find the general solution of $y'' - 2y' + 10y = 26e^{-t}$.

Since e^{-t} is not a solution of the homogeneous equation, we guess $y_p = Ae^{-t}$. Then $y_p' = -Ae^{-t}$ and $y_p'' = Ae^{-t}$ and so

$$y_p'' - 2y_p' + 10y_p = 13Ae^{-t}$$

which is supposed to be $26e^{-t}$, so $A = 2$ The general solution is

$$y = 2e^{-t} + c_1e^t\cos 3t + c_2e^t\sin 3t$$

(c) Solve the initial-value problem $y'' - 2y' + 10y = 26e^{-t}$, $y(0) = 7$, $y'(0) = 0$.

From the general solution in part (b), we have $y(0) = 2 + c_1$ and $y'(0) = -2 + c_1 + 3c_2$. So we need to solve

$$\begin{aligned} c_1 + 2 &= 7 \\ c_1 + 3c_2 - 2 &= 0 \end{aligned}$$

and we get $c_1 = 5$ and $c_2 = -1$. The (unique) solution of the initial-value problem is

$$y = 2e^{-t} + 5e^t\cos 3t - e^t\sin 3t$$

6. (a) Find the general solution of the equation $y'' + 2y' + y = 0$.

The roots of the auxiliary equation are -1 and -1 so the solution is $y = c_1e^{-t} + c_2te^{-t}$

(b) Now find the general solution of $y'' + 2y' + y = 18e^{2t}$.

Undetermined coefficients: Guess $y_p = Ae^{2t}$, then $y_p' = 2Ae^{2t}$ and $y_p'' = 4Ae^{2t}$, therefore $y_p'' + 2y_p' + y_p = 9Ae^{2t}$ so $A = 2$. Conclude

$$y = 2e^{2t} + c_1e^{-t} + c_2te^{-t}$$

(c) Finally, find the general solution of $y'' + 2y' + y = \dfrac{e^{-t}}{t}$

Variation of parameters: we have $W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (-t+1)e^{-t} \end{vmatrix} = e^{-2t}$, so the particular solution is

$$y_p = e^{-t}\int \frac{-te^{-t}\,e^{-t}}{te^{-2t}}\,dt + te^{-t}\int \frac{e^{-t}\,e^{-t}}{te^{-2t}}\,dt = -te^{-t} + te^{-t}\ln t$$

Therefore the general solution is

$$y = -te^{-t} + te^{-t}\ln t + c_1e^{-t} + c_2te^{-t}$$

Note, you could absorb the first term into the c_2 term.

6. (a) Find the general solution of the equation $y'' - 2y' + y = 0$.

$r^2 - 2r + 1 = (r-1)^2$ so the general solution is $y = c_1e^t + c_2te^t$.

(b) Find the Wronskian of the two independent solutions of $y'' - 2y' + y = 0$ that you found in part (a).

With $y_1 = e^t$ and $y_2 = te^t$, $W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$.

(c) Find the general solution of the equation $y'' - 2y' + y = \dfrac{e^t}{t+4}$.

Variation of parameters.

$$\begin{aligned} y &= y_1\int \frac{-y_2 f}{W}\,dt + y_2\int \frac{y_1 f}{W}\,dt \\ &= e^t\int \frac{-te^t}{e^{2t}}\frac{e^t}{t+4}\,dt + te^t\int \frac{e^t}{e^{2t}}\frac{e^t}{t+4}\,dt \\ &= e^t\int \frac{-t}{t+4}\,dt + te^t\int \frac{1}{t+4}\,dt \\ &= e^t\int \frac{4}{t+4} - 1\,dt + te^t(\ln(t+4) + c_2) \\ &= e^t(4\ln(t+4) - t + c_1) + te^t\ln(t+4) + c_2te^t \\ &= 4e^t\ln(t+4) - te^t + te^t\ln(t+4) + c_1e^t + c_2te^t \end{aligned}$$

(You could absorb the $-te^t$ into the c_2 term and simplify to $y = (t+4)e^t\ln(t+4) + c_1e^t + c_2te^t$)

3. The function $y_1(t) = t$ is a solution of $(t^2-2)y'' - 2ty' + 2y = 0$. Find the general solution of this equation.

Two methods: The Wronskian W of two independent solutions of the equation satisfies $-\frac{W'}{W} = -\frac{2t}{t^2-2}$. Therefore $\ln W = \ln(t^2-2)$, so $W = t^2-2$. Therefore

$$W = \begin{vmatrix} t & y_2 \\ 1 & y_2' \end{vmatrix} = ty' - y = t^2 - 2$$

so y_2 satisfies the first-order equation $y_2' - \frac{1}{t}y_2 = t - \frac{2}{t}$. The integrating factor is $1/t$, so $\left(\frac{y_2}{t}\right)' = 1 - \frac{2}{t^2}$ and we have

$$\frac{y_2}{t} = t + \frac{2}{t}$$

so

$$y_2 = t^2 + 2$$

Therefore the general solution is:

$$y = c_1 t + c_2(t^2+2)$$

The other way: Let $y_2 = tv$. Then $y_2' = tv' + v$ and $y_2'' = tv'' + 2v'$ and the differential equation becomes:

$$(t^2-2)(tv''+2v') - 2t(tv'+v) + 2tv = 0$$

or

$$t(t^2-2)v'' + -4v' = 0 \implies \ln(v') = \int \frac{4}{t(t^2-2)}\,dt = \int \frac{4t}{t^2(t^2-2)}\,dt = \int \frac{1}{t^2-2} - \frac{1}{t^2}\,dt$$

So $v' = 1 - \frac{2}{t^2}$ and $v = t + \frac{2}{t}$, and $y_2 = tv = t^2 + 2$ as above.

3. The function $y_1(t) = e^t$ is a solution of $ty'' - 2y' + (2-t)y = 0$. Find the general solution of this equation.

Two methods: The Wronskian W of two independent solutions of the equation satisfies $-\frac{W'}{W} = -\frac{2}{t}$. Therefore $\ln W = \ln t^2$, so $W = t^2$. Therefore

$$W = \begin{vmatrix} e^t & y_2 \\ e^t & y_2' \end{vmatrix} = e^t(y' - y) = t^2$$

so y_2 satisfies the first-order equation $y' - y = t^2 e^{-t}$. The integrating factor is e^{-t}, so $\left(y_2 e^{-t}\right)' = t^2 e^{-2t}$ and we have (after integrating by parts)

$$y_2 e^{-t} = -\frac{1}{4}(2t^2 + 2t + 1)e^{-2t} + C$$

so (with $C = 0$)

$$y_2 = -\frac{1}{4}(2t^2 + 2t + 1)e^{-t}$$

Therefore the general solution is:

$$y = c_1 e^t + c_2(2t^2 + 2t + 1)e^{-t}$$

(where we've absorbed the $-\frac{1}{4}$ into c_2).

The other way: Let $y_2 = e^t v$. Then $y_2' = e^t v' + e^t v$ and $y_2'' = e^t v'' + 2e^t v' + e^t v$ and the differential equation becomes:

$$e^t\Big(t(v'' + 2v' + v) - 2(v' + v) + (2-t)v\Big) = 0$$

or

$$tv'' + (2t-2)v' = 0 \implies \ln(v') = \int \frac{2}{t} - 2\,dt = \ln t^2 - 2t$$

So $v' = t^2 e^{-2t}$ and (after that same integration by parts) $v = -\frac{1}{4}(2t^2 + 2t + 1)e^{-2t}$. Therefore $y_2 = e^t v = -\frac{1}{4}(2t^2 + 2t + 1)e^{-t}$ as above.

4. Find the general solution to the ordinary differential equation:

$$f'''(t) - 2f''(t) + f'(t) = 4t + 2e^{2t}.$$

Solution First find the homogeneous solution. The characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0.$$

We factor and obtain the roots.

$$\lambda(\lambda^2 - 2\lambda + 1) = \lambda(\lambda - 1)^2 = 0 \implies \lambda = 0,\ 1 \text{ (mult 2)}$$

This generates the following general solution of for the homogeneous equation

$$C_1e^{0t} + C_2e^{1t} + C_3te^{1t} = C_1 + C_2e^t + C_3te^t$$

We guess a solution to the non-homogeneous equation. The second term as an exponential dictates a guess of the form Ae^{2t}, and the first term is a degree one polynomial which requires a guess of the form $Bt + C$. The constant term is in the homogeneous solution so instead we must guess $t(Bt + C) = Bt^2 + Ct$. Due to the linearity we may just add the guesses for each term. Therefore our particular solution will have the form

$$f_p(t) = Ae^{2t} + Bt^2 + Ct,$$

From which we see that

$$f_p'(t) = 2Ae^{2t} + 2Bt + C, \quad f_p''(t) = 4Ae^{2t} + 2B, \quad f_p'''(t) = 8Ae^{2t}.$$

Substituting into the ODE gives

$$8Ae^{2t} - 8Ae^{2t} - 4B + 2Ae^{2t} + 2Bt + C = 4t + 2e^{2t}.$$

Equating coefficients of like functions gives the system

$$\begin{cases} 2A = 2 \\ 2B = 4 \\ C - 4B = 0 \end{cases}$$

which has the solution $A = 1, B = 2, C = 8$. The any solution to the non-homogeneous ODE can be written as a particular solution plus some homogeneous solution. Therefore the general solution is

$$\boxed{C_1 + C_2e^t + C_3te^t + e^{2t} + 2t^2 + 8t}$$

1. (a) Find the general solution of $y'''' - 3y'' - 4y = 0$.

The auxiliary polynomial is $r^4 - 3r^2 - 4 = (r^2 - 4)(r^2 + 1) = (r + 2)(r - 2)(r^2 + 1)$. So the solution is

$$y = c_1e^{-2t} + c_2e^{2t} + c_3 \cos t + c_4 \sin t$$

(b) Find the general solution of $y'''' - 3y'' - 4y = 20t$

From part (a) we need to guess $y_p = At + B$. Then $y_p' = A$ and $y_p'' = y_p''' = y_p'''' = 0$. Therefore

$$y_p'''' - 3y_p'' - 4y_p = -4(At + B)$$

So we need $A = -5$ and $B = 0$, so

$$y = -5t + c_1e^{-2t} + c_2e^{2t} + c_3 \cos t + c_4 \sin t$$

5. An undamped mass/spring system satisfies the differential equation $y'' + 16y = 0$.

(a) If the mass is released from rest, 0.2 meters from its equilibrium point, how long until it returns to its starting point (for the first time after being released)?

The motion will satisfy $y(t) = 0.2\cos(4t)$. The mass will return to 0.2 meters when $4t = 2\pi$, or $\pi/2$ seconds later.

(b) If the mass is released from rest, 0.4 meters from its equilibrium point, now how long until it returns to its starting point (for the first time)?

Now the motion will satisfy $y(t) = 0.4\cos(4t)$, so the answer is still $\pi/2$ seconds later.

(c) Now suppose the mass is doubled and it is released from rest, 0.2 meters from its equilibrium point. Now how long until it returns to its starting point?

This changes the differential equation to $2y'' + 16y = 0$, or $y'' + 8y = 0$. so the solution is $y = 0.2\cos(\sqrt{8}\,t)$, so the mass returns to the starting point when $\sqrt{8}\,t = 2\pi$, or $\pi/\sqrt{2}$ seconds later.

1. (a) Use the Wronskian to show that the functions $y_1 = t$ and $y_2 = e^t$ are linearly independent.

We have $W = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = (t-1)e^t \neq 0$, so the functions are linearly independent.

(b) Find a homogeneous linear second-order differential equation having the functions from part (a), $y_1 = t$ and $y_2 = e^t$, as solutions.

We want $y'' + py' + qy = 0$, where $p = -W'/W = -\frac{te^t}{(t-1)e^t} = -\frac{t}{t-1}$. So far, the equation reads

$$y'' - \frac{t}{t-1}y' + qy = 0.$$

To find q, we'll substitute $y_1 = e^t$ for y and get

$$e^t\left(1 - \frac{t}{t-1} + q\right) = 0,$$

so $q = \frac{1}{t-1}$. So the differential equation is

$$y'' - \frac{t}{t-1}y' + \frac{1}{t-1}y = 0 \quad \text{or} \quad (t-1)y'' - ty' + y = 0$$

Check by substituting $y = t$.

(c) Find a third-order differential equation *with constant coefficients* for which $y_1 = t$ and $y_2 = e^t$ are two linearly independent solutions. (What must the other one be?)

Since t is a generalized eigenfunction of D with eigenvalue $\lambda = 0$, the other solution must be $y = 1$. Therefore the equation is

$$D^2(D-1)y = 0 \qquad \text{or} \qquad y''' - y'' = 0$$

8. Suppose $y(t)$ is the displacement of a door to the U.S. Capitol at time t. As a result of the strength of the door and rioters pushing on it, $y(t)$ satisfies the differential equation

$$y''(t) = -y(t) + f(t)$$

where $f(t)$ results from the force applied by the rioters. A quick-thinking Capitol police officer realizes three things:

- $y(0) = 0$ and $y'(0) = 1$.
- The door will break when $y(t) = 1$.
- Reinforcements won't arrive till $t = 1$. For $0 \leqslant t \leqslant 1$ the rioters will maintain $f(t) = 1$. It's not clear what $f(t)$ will be for $t \geqslant 1$.

Fortunately, the officer took Math 2400. From the above information, can she decide whether the door will break before $t = 1$? If so, will it break or not before $t = 1$? Justify your answers.

Again, looking past the blather, the problem is asking whether or not the solution $y(t)$ of the initial-value problem $y'' + y = 1$, $y(0) = 0$, $y'(0) = 1$ will be bigger than 1 for any value of t between 0 and 1. It's not hard to see (by undetermined coefficients) that the general solution of $y'' + y = 1$ is $y = 1 + c_1 \cos t + c_2 \sin t$. For $y(0) = 0$ we need $c_1 = -1$ and for $y'(0) = 1$ we need $c_2 = 1$. So the solution is

$$y(t) = 1 - \cos t + \sin t.$$

We'll have $y(t) = 1$ for $t = \pi/4$, which is less than 1. So it looks like the door will break.

4. Suppose that $y(t)$ is the height above the ground at time t of a miniature helicopter carried underneath the carriage of a Martian rover. When the rover touches down at time $t = 0$, one has $y(0) = .4$ meters. The rover bounces when it lands. This leads to $y'(0) = -1$ meters per second and the differential equation

$$y''(t) = -2(y(t) - .4) - 2y'(t).$$

The helicopter will be damaged if $y(t)$ ever becomes negative for some $t > 0$. Will this happen? Justify your answer.

Hints: Minimize the function $y(t)$. Remember that $y'(t) = 0$ when $y(t)$ is minimal. You can use the fact that

$$\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2} < 0.4 \cdot e^{\pi/4}.$$

In standard form, the differential equation is $y'' + 2y' + 2y = 0.8$. The auxiliary equation is $r^2 + 2r + 2 = 0$, with solution $r = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. So the complementary solution is $y_c = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$ and (by undetermined coefficients) the particular solution is $y_p = 0.4$. The general solution is thus

$$y = 0.4 + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

For $y(0) = 0.4$, we'll need $c_1 = 0$. Then

$$y' = c_2 e^{-t}(\cos t - \sin t)$$

and $y'(0) = -1$ means $c_2 = -1$. Thus $y(t) = 0.4 - e^{-t}\sin t$. From the expression for y' above, we'll have $y' = 0$ when $\cos t = \sin t$, so for instance when $t = \pi/4$. And $y(\pi/4) = 0.4 - e^{-\pi/4}\sqrt{2}/2$. By the hint, this quantity is bigger than zero, so the helicopter will **not** be damaged.

10.1: Springs

4. (a) The following two differential equations describe the free motion of a mass/spring system. Are the systems overdamped, underdamped or critically damped? Explain briefly how you know.

(i) $y'' + 4y' + 10y = 0$

The roots of the auxiliary equation are $\frac{1}{2}(-4 \pm \sqrt{16-40}) = -2 \pm i\sqrt{24}$ – negative real part and nonzero imaginary part, so this system is underdamped.

(ii) $y'' + 10y' + 4y = 0$

The roots of the auxiliary equation are $\frac{1}{2}(-10 \pm \sqrt{100-16}) = -5 \pm \frac{1}{2}\sqrt{84}$ – both real and negative (and different), so this system is overdamped.

(b) The following two differential equations describe the forced motion of a mass/spring system. Do the systems exhibit modulation, transience, or resonance?

(i) $y'' + 4y' + 10y = 16\cos t$

From (a)(i), the natural vibrations of the system die out, so this system experiences transience.

(ii) $y'' + 121y = 3\sin 11t$

The natural vibrations have the same frequency as the forcing function, so this system exhibits reasonance.

4. (a) The following two differential equations describe the free motion of a mass/spring system. Are the systems overdamped, underdamped or critically damped? Explain briefly how you know.

(i) $y'' + 4y' + 20y = 0$

The roots of the auxiliary equation are $\frac{1}{2}(-4 \pm \sqrt{16-80}) = -2 \pm 8i$ – negative real part and nonzero imaginary part, so this system is underdamped.

(ii) $y'' + 20y' + 4y = 0$

The roots of the auxiliary equation are $\frac{1}{2}(-20 \pm \sqrt{400-16}) = -10 \pm \frac{1}{2}\sqrt{384} = -10 \pm 4\sqrt{6}$ – both real and negative (and different), so this system is overdamped.

(b) The following two differential equations describe the forced motion of a mass/spring system. Do the systems exhibit modulation, transience, or resonance?

(i) $y'' + 4y' + 20y = 7\cos 3t$

From (a)(i), the natural vibrations of the system die out, so this system experiences transience.

(ii) $y'' + 81y = 12\sin 9t$

The natural vibrations have the same frequency as the forcing function, so this system exhibits resonance.

10.2: Cauchy-Euler Equations

2. (a) Find the general solution $y(x)$ of

$$3x^2y'' + 10xy' - 6y = 0$$

The auxiliary equation is $3p(p-1) + 10p - 6 = 3p^2 + 7p - 6 = (3p-2)(p+3) = 0$ so the general solution is

$$y = c_1x^{2/3} + c_2x^{-3}$$

(b) What is the unique solution (for $x > 0$) of $3x^2y'' + 10xy' - 6y = 0$ for which $y(1) = 2$ and $y'(1) = -6$?

If $y(1) = 2$ then $c_1 + c_2 = 2$. Since $y' = \frac{2}{3}c_1x^{-1/3} - 3c_2x^{-4}$, we also have $\frac{2}{3}c_1 - 3c_2 = -6$. Therefore $c_1 = 0$ and $c_2 = 2$ and the solution is

$$y = \frac{2}{x^3}$$

1. (a) Find the general solution of the differential equation $x^2y'' + 5xy' + 4y = 0$.

Guess that $y = x^p$, then $y' = px^{p-1}$ and $y'' = p(p-1)x^{p-2}$. Substitute into the equation and get $x^p\Big(p(p-1) + 5p + 4\Big) = 0$, so $p^2 + 4p + 4 = (p+2)^2 = 0$. Since the root is repeated the solution is

$$y = c_1x^{-2} + c_2x^{-2}\ln x$$

(b) Find the unique solution of the equation in part (a) for which $y(1) = 2$ and $y'(1) = 5$.

$y(1) = c_1$, so we need $c_1 = 2$. $y' = -2c_1x^{-3} - 2c_2x^{-3}\ln x + c_2x^{-3}$, so $y'(1) = -2c_1 + c_2$ therefore $c_2 = 9$. The solution is

$$y = 2x^{-2} + 9x^{-2}\ln x$$

11.1: Systems of Differential Equations — basics

5. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(a) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$

The characteristic polynomial of A is $(-\lambda)(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.

For $\lambda = 2$, $A - 2\mathbf{I} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, $A + \mathbf{I} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = c_1e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) The origin is the only critical point of the system $\mathbf{x}' = A\mathbf{x}$. What kind of critical point is it?

Since the eigenvalues of A are real and have opposite signs, the origin is a saddle point.

4. (a) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$.

The characteristic polynomial of A is $(1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$ so the eigenvalues of A are 3 and -2.

For $\lambda = 3$ we have $A - 3\mathbf{I} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

For $\lambda = -2$ we have $A + 2\mathbf{I} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Therefore, the general solution of the system is

$$\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b) Find the unique solution of the system in part (a) for which $\mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

We need $3c_1 - c_2 = 10$ and $2c_1 + c_2 = 0$, so $c_2 = -2c_1$ and then $5c_1 = 10$, so $c_1 = 2$ and $c_2 = -4$. Thus

$$\mathbf{x}(t) = 2e^{3t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

4. (a) Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Expanding the determinant along the second row, the characteristic polynomial of A is $(2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = -\lambda(2-\lambda)(3-\lambda)$, so the eigenvalues of A are $\lambda = 0$, $\lambda = 2$ and $\lambda = 3$.

For $\lambda = 0$, $A - 0\mathbf{I} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

For $\lambda = 2$, $A - 2\mathbf{I} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$ and the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda = 3$, $A - 3\mathbf{I} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$ and the eigenvector is $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

The general solution is $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

(b) For the matrix A given above, find the unique solution of the initial-value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$.

To match these initial conditions we need $c_1 = -1$, $c_2 = 1$ and $c_3 = 1$. Thus

$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

11.2: Constant coefficient systems

2. Consider the third-order differential equation

$$y''' + 3y'' + 2y' = 12e^t$$

(a) Write an equivalent first-order system of three equations in three unknown functions in the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ for a matrix of constants A and a vector of functions $\mathbf{f}$.

(b) Solve either the third-order equation or the system and then write the general solution of the equation.

(a) The equivalent system (with $x_1 = y$) is $x_1' = x_2$, $x_2' = x_3$ and $x_3' = -2x_2 - 3x_3$, so

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ 12e^t \end{bmatrix}$$

(b) It's probably easier to solve the equation: the auxiliary polynomial is $r^3 + 3r^2 + 2r = r(r + 1)(r + 2)$ so the complementary solution is

$$y_c = c_1 + c_2e^{-t} + c_3e^{-2t}$$

For the particular solution, guess $y = Ae^t$, and then $y''' + 3y'' + 2y'' = 6Ae^t$, so $A = 2$ and the general solution of the equation is

$$y = y_p + y_c = 2e^t + c_1 + c_2e^{-t} + c_3e^{-2t}$$

3. Consider the following first-order system of equations:

$$\mathbf{x}' = A\mathbf{x} \qquad \text{where} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & -13 & 0 \end{bmatrix}$$

(a) Write an equivalent fourth-order equation for a single unknown function.

(b) Solve either the system or the equation and then write the general solution of the first-order system.

(a) The equivalent equation is $y'''' = -36y - 13y''$, or $y'''' + 13y'' + 36y = 0$.

(b) The auxiliary equation is $r^4 + 13r^2 + 36 = (r^2 + 4)(r^2 + 9) = 0$. The general solution of the fourth-order equation is thus

$$y = c_1 \cos 2t + c_2 \sin 2t + c_3 \cos 3t + c_4 \sin 3t$$

and the solution of the system is thus

$$\mathbf{x} = \begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = c_1 \begin{bmatrix} \cos 2t \\ -2\sin 2t \\ -4\cos 2t \\ 8\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2\cos 2t \\ -4\sin 2t \\ -8\cos 2t \end{bmatrix} + c_3 \begin{bmatrix} \cos 3t \\ -3\sin 3t \\ -9\cos 3t \\ 27\sin 3t \end{bmatrix} + c_4 \begin{bmatrix} \sin 3t \\ 3\cos 3t \\ -9\sin 3t \\ -27\cos 3t \end{bmatrix}$$

1. (a) Show that the functions $y_1 = te^{-2t}$ and $y_2 = \cos t$ are linearly independent.

Calculate the Wronskian: $W = \begin{vmatrix} te^{-2t} & \cos t \\ (1-2t)e^{-2t} & -\sin t \end{vmatrix} = (2t-1)e^{-2t}\cos t - te^{-2t}\sin t \neq 0$ (observe that $W(0) = -1$) so y_1 and y_2 are linearly independent.

(b) Write a fourth-order homogeneous differential equation with constant coefficients that has $y_1 = te^{-2t}$ and $y_2 = \cos t$ as solutions.

Two other linearly independent solutions of the equation must be e^{-2t} and $\sin t$, so the auxiliary polynomial is $(r+2)^2(r^2+1) = r^4 + 4r^3 + 5r^2 + 4r + 4$. The equation is thus

$$y'''' + 4y''' + 5y'' + 4y' + 4y = 0.$$

(c) Write a system of the form $\mathbf{x}' = A\mathbf{x}$ with a constant 4-by-4 matrix A that is equivalent to the equation you found in part (b).

We'll let $x_1 = y$, $x_2 = y'$, $x_3 = y''$ and $x_4 = y'''$. Then the system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -4x_1 - 4x_2 - 5x_3 - 4x_4 \end{aligned}$$

and so $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -4 & -5 & -4 \end{bmatrix}.$

2. Consider the following system of second-order equations:

$$\begin{aligned} x'' + 3x' + 4y' - 2z &= \cos t \\ y'' - 4x - y + 2y' + 3z &= \sin t \\ z'' + 2x' + 3y + 4z + z' &= 0 \end{aligned}$$

Find a system of first-order equations to which this system is equivalent, and write it in vector/matrix form. (You do not have to solve the system).

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$, $x_5 = z$ and $x_6 = z'$. The system is equivalent to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & -4 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -3 & 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \\ 0 \\ \sin t \\ 0 \\ 0 \end{bmatrix}$$

3. Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 9 \\ -1 & 1 \end{bmatrix}$

The characteristic polynomial of A is $(1-\lambda)(1-\lambda)+9 = \lambda^2 - 2\lambda + 10$, so the eigenvalues of A are $\frac{1}{2}(2 \pm \sqrt{4-40}) = 1 \pm 3i$.

$A - (1+3i)\mathbf{I} = \begin{bmatrix} -3i & 9 \\ -1 & -3i \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 3 \\ i \end{bmatrix}$. And we have

$$e^{(1+3i)t}\begin{bmatrix} 3 \\ i \end{bmatrix} = e^t(\cos 3t + i\sin 3t)\begin{bmatrix} 3 \\ i \end{bmatrix} = e^t\begin{bmatrix} 3\cos 3t \\ -\sin 3t \end{bmatrix} + ie^t\begin{bmatrix} 3\sin 3t \\ \cos 3t \end{bmatrix}$$

So the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1e^t\begin{bmatrix} 3\cos 3t \\ -\sin 3t \end{bmatrix} + c_2e^t\begin{bmatrix} 3\sin 3t \\ \cos 3t \end{bmatrix}$$

If instead you use the eigenvector $\begin{bmatrix} 3i \\ -1 \end{bmatrix}$ then you get

$$e^{(1+3i)t}\begin{bmatrix} 3i \\ -1 \end{bmatrix} = e^t(\cos 3t + i\sin 3t)\begin{bmatrix} 3i \\ -1 \end{bmatrix} = e^t\begin{bmatrix} -3\sin 3t \\ -\cos 3t \end{bmatrix} + ie^t\begin{bmatrix} 3\cos 3t \\ -\sin 3t \end{bmatrix}$$

So the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1e^t\begin{bmatrix} 3\sin 3t \\ \cos 3t \end{bmatrix} + c_2e^t\begin{bmatrix} 3\cos 3t \\ -\sin 3t \end{bmatrix}$$

(which is the same).

12.1: Non-diagonalizable coefficient matrices

3. Let $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$.

(a) Find the Jordan canonical form J of A.

The eigenvalues of A are $\lambda = 4$ and $\lambda = -1$ (with multiplicity 3).

For $\lambda = 4$, we have $A - 4\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 1 & -1 & 1 & -5 \end{bmatrix}$, so an eigenvector is $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, we have $A + \mathbf{I} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$, which has rank 2, so there are only two linearly independent eigenvectors, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(b) Find a matrix P such that $A = PJP^{-1}$ (or, equivalently, $J = P^{-1}AP$).

We have the eigenvectors for $\lambda = 4$ and $\lambda = -1$, and we need a generalized eigenvector for $\lambda = -1$.

Since $(A + \mathbf{I})^2 = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}_2 = (A + \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and so $P = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

(c) Find the general solution of the homogeneous system $\mathbf{y}' = J\mathbf{y}$, where J is the Jordan form of A that you found in part (a) (this is equivalent to calculating e^{tJ}).

The solution of $\mathbf{y}' = J\mathbf{y}$ is $\mathbf{y} = c_1 e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} 0 \\ t \\ 1 \\ 0 \end{bmatrix} + c_4 e^{-t}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Or, you could say that $e^{tJ} = \begin{bmatrix} e^{4t} & 0 & 0 & 0 \\ 0 & e^{-t} & te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$

(d) Using your answer to parts (a), (b) and (c), write the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$.

The solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = P\mathbf{y} = c_1 e^{4t}\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -t \end{bmatrix} + c_4 e^{-t}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

parts (a) and (b) are also listed in 7.1

12.2: Non-homogeneous systems

5. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(a) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$

The characteristic polynomial of A is $(-\lambda)(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.
For $\lambda = 2$, $A - 2\mathbf{I} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
For $\lambda = -1$, $A + \mathbf{I} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
The general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) The origin is the only critical point of the system $\mathbf{x}' = A\mathbf{x}$. What kind of critical point is it?

Since the eigenvalues of A are real and have opposite signs, the origin is a saddle point.

(c) Find the general solution of the system $\mathbf{x}' = A\mathbf{x} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Undetermined coefficients: Guess $\mathbf{x}_p = e^t \mathbf{a}$. Then $e^t \mathbf{a} = e^t A\mathbf{a} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so we need $\mathbf{a}$ to satisfy $(\mathbf{I} - A)\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The linear system $\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -1 & 0 & 1 \end{array}\right]$ has solution $\mathbf{a} = \begin{bmatrix} -1 \\ -3/2 \end{bmatrix}$, so $\mathbf{x}_p = -\frac{1}{2} e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$$\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \frac{1}{2} e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Variation of parameters: From (a) we have a solution matrix for the homogeneous system: $X = \begin{bmatrix} e^{2t} & -2e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}$, with $\det(X) = 3e^t$ and $X^{-1} = \frac{1}{3} e^{-t} \begin{bmatrix} e^{-t} & 2e^{-t} \\ -e^{2t} & e^{2t} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-2t} & 2e^{-2t} \\ -e^t & e^t \end{bmatrix}$. So we have

$$\mathbf{x}_p = X \int X^{-1} \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} = X \int \frac{1}{3} \begin{bmatrix} 4e^{-t} \\ -e^{2t} \end{bmatrix} = X \begin{bmatrix} -\frac{4}{3} e^{-t} \\ -\frac{1}{6} e^{2t} \end{bmatrix} = \begin{bmatrix} -e^t \\ -\frac{3}{2} e^t \end{bmatrix}$$

which agrees with the undetermined coefficients version.

parts (a) and (b) are also listed in 11.1

2. (a) Find the fundamental solution matrix of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. In other words, calculate e^{tA}

A is already in Jordan canonical form, with eigenvalue $\lambda = 1$, so its fundamental solution matrix is $X(t) = e^{tA} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$.

(b) Now find the general solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, for the matrix A in part (a) and the vector $\mathbf{f} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Undetermined coefficients: Guess $\mathbf{x} = \mathbf{a}$, a constant vector. Substituting into the equation shows that we need $\mathbf{0} = A\mathbf{a} + \mathbf{f}$, so $\mathbf{a} = -A^{-1}\mathbf{f} = -\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. So the general solution is

$$\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \end{bmatrix}$$

or

Variation of Parameters: The Wronskian of our two solutions is the determinant of e^{tA}, so $W = e^{2t}$. Then

$$\begin{aligned} \mathbf{x}_p = X(t)\int X(-t)\mathbf{f}\,dt &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \int \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} dt \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \int \begin{bmatrix} 2e^{-t} - te^{-t} \\ e^{-t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -e^{-t} + te^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \end{aligned}$$

as before.

(c) Find the general solution of the equation $y'' - 2y' + y = \dfrac{e^t}{t+4}$.

Variation of parameters.

$$\begin{aligned} y &= y_1 \int \frac{-y_2 f}{W}\,dt + y_2 \int \frac{y_1 f}{W}\,dt \\ &= e^t \int \frac{-te^t}{e^{2t}} \frac{e^t}{t+4}\,dt + te^t \int \frac{e^t}{e^{2t}} \frac{e^t}{t+4}\,dt \\ &= e^t \int \frac{-t}{t+4}\,dt + te^t \int \frac{1}{t+4}\,dt \\ &= e^t \int \frac{4}{t+4} - 1\,dt + te^t(\ln(t+4) + c_2) \\ &= e^t(4\ln(t+4) - t + c_1) + te^t\ln(t+4) + c_2te^t \\ &= 4e^t\ln(t+4) - te^t + te^t\ln(t+4) + c_1e^t + c_2te^t \end{aligned}$$

(You could absorb the $-te^t$ into the c_2 term and simplify to $y = (t+4)e^t\ln(t+4) + c_1e^t + c_2te^t$)

13.1: The geometry of systems of differential equations

5. Consider systems of the form $x' = Ax$ where A is a matrix of the form

$$\begin{bmatrix} 2 & 1 \\ a^2 - 1 & 0 \end{bmatrix}$$

in which a can be any real number.

(a) For what values of a, if any, do all trajectories flow towards the origin (i.e., $\lim_{t\to\infty} x(t) = (0,0)$ for all trajectories, a.k.a. solutions to they system, x)?

The eigenvalues of A are the roots of $(2-\lambda)(-\lambda) - (a^2-1) = 0$, i.e., $\lambda^2 - 2\lambda + 1 - a^2 = 0$, so the roots are

$$\lambda = \frac{2 \pm \sqrt{4 - 4 + 4a^2}}{2} = 1 \pm a$$

Thus, both eigenvalues are real, and for any value of a at least one of them is positive, so there are **no** values of a for which all the trajectories flow toward the origin.

In fact, for $-1 < a < 1$, both eigenvalues are positive and the origin is an unstable node. If $a > 1$ or $a < -1$, then there is an eigenvalue of each sign and the origin is a saddle point, and if $a = \pm 1$ then every point on the line $y = -2x$ will be a critical (equilibrium) point. In this case, all the trajectories on this line will neither flow towards or out of the origin.

(b) For what values of a, if any, do all trajectories flow out of the origin (i.e., $\lim_{t\to-\infty} x(t) = (0,0)$ for all trajectories, a.k.a. solutions to they system, x)?

If $-1 < a < 1$ then both eigenvalues are positive, so the origin will be an unstable node .

(c) For what values of a, if any, do some trajectories flow out of the origin and all other trajectories flow towards the origin?

By the above analysis, there are no such values of a.

13.2: Nonlinear autonomous systems of differential equations

3. (a) Find the general solution of the homogeneous linear system: $\begin{cases} x'(t) = 4x(t) - 2y(t) \\ y'(t) = -2x(t) + 4y(t) \end{cases}$

(b) Determine which figure below represents the trajectories of the solutions of the system. You must justify your answer.

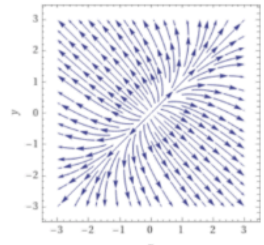


Figure 1

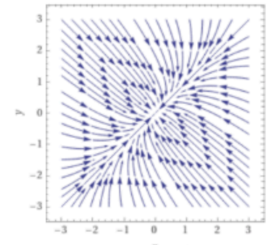


Figure 2

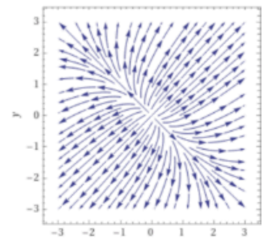


Figure 3

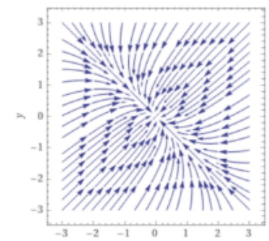


Figure 4

Solutions

(a) This system is of the form $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, where $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$. The rows add up to 2 thus $\lambda_1 = 2$ is an eigenvalue of A with eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since the trace is sum of eigenvalues, the other eigenvalue is $\lambda_2 = 6$. Since A is symmetric, the eigenvector v_2 of λ_2 is orthogonal to v_1. We can pick $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus general solution of the system is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) Since the eigenvalues are distinct and positive, the trajectories form a source node, and as v_2 has the largest eigenvalue, the trajectories are parallel to v_2 overtime, thus is must be $\boxed{\text{Figure 1}}$.

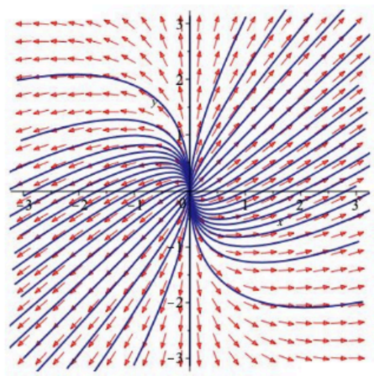
7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since the matrix A is lower triangular, its eigenvalues are $\lambda = 1$ and $\lambda = 2$. Therefore the origin is an unstable node (improper source node).

(b) Sketch the phase portrait for this system, paying particular attention to straight-line trajectories (if any), and to how (and in what direction) the other trajectories go as $t \to \pm\infty$.

We need the eigenvectors of A. For $\lambda = 1$, an eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and for $\lambda = 2$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The trajectories will emanate from the origin, tangent to the y-axis (i.e., in the direction of the eigenvector for the smaller eigenvalue, $\lambda = 1$) and curve so they become parallel to the line $y = x$ (parallel to the eigenvector for the larger eigenvalue) as $t \to \infty$. The straight-line trajectories are the positive and negative y-axis and the parts of the line $y = x$ in the first and third quadrants.



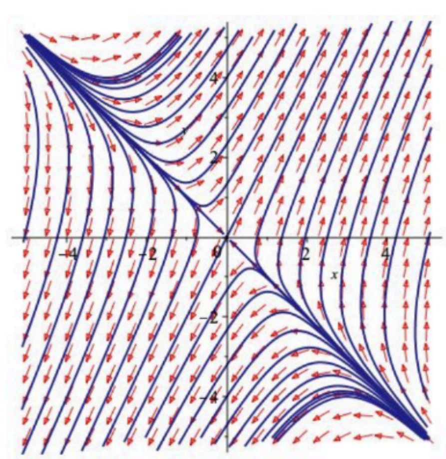
7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(a) Of course, the origin is the only critical point of this system. What kind of critical point is it?

Since $\det(A) = -5$, the origin is a saddle point. Also, the sum of both columns is 5, so that is one of the eigenvalues of A, and the other is -1, since the trace of A is 4.

(b) Sketch the phase portrait for the system $\mathbf{x}' = A\mathbf{x}$, drawing several representative trajectories and indicating their direction.

The eigenvector for $\lambda = 5$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ – trajectories go out along this vector (and its negative). The eigenvector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and trajectories come in along this vector.



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = (3 - x)(y + 4)$$

$$\frac{dy}{dt} = x(1 - y)$$

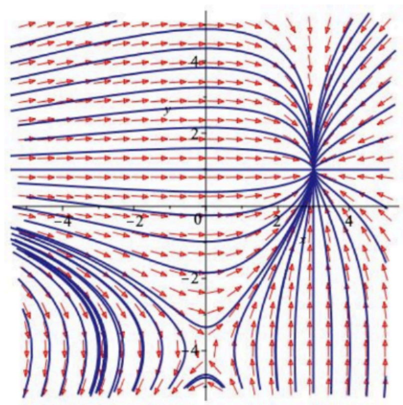
For the critical points, we need both right-hand sides to be zero. The first one is zero if $x = 3$ or $y = -4$ and the second if $x = 0$ or $y = 1$. Therefore there are two critical points $(3, 1)$ and $(0, -4)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} -y-4 & 3-x \\ 1-y & -x \end{bmatrix}$.

At $(3, 1)$ we have $J(3, 1) = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}$, which is a stable node (or improper sink).

At $(0, -4)$, $J(0, -4) = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}$, which is a saddle.

For the record:



8. Find and classify all of the critical (equlibrium) points of the nonlinear system

$$\frac{dx}{dt} = (2 - x)(4 - y)$$

$$\frac{dy}{dt} = x(1 + y)$$

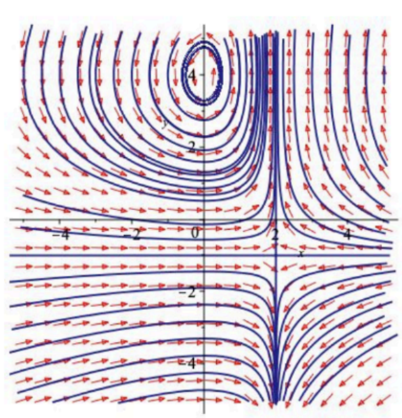
For the critical points, we need both right-hand sides to be zero. The first one is zero if $x = 2$ or $y = 4$ and the second if $x = 0$ or $y = -1$. Therefore there are two critical points $(2, -1)$ and $(0, 4)$.

The Jacobian of the system is $J(x, y) = \begin{bmatrix} y-4 & x-2 \\ y+1 & x \end{bmatrix}$.

At $(2, -1)$ we have $J(2, -1) = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$, which has negative determinant so $(2, -1)$ is a saddle point.

At $(0, 4)$, $J(0, 4) = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}$, which is a center.

For the record:



6. Consider a system

$$x' = 2x(x-2)(y+3)$$
$$y' = (x+2)(y+4)$$

Find all critical points and describe the behavior of the system near these critical points.

For $x' = 0$ we need either $x = 0$, $x = 2$ or $y = -3$. If $x = 0$ or $x = -2$, then the second equation says $y = -4$. If $y = -3$, then the second equation says $x = -2$. Thus there are three critical points: $(0, -4)$, $(2, -4)$ and $(-2, -3)$.

The Jacobian of the system is

$$J = \begin{bmatrix} (4x-4)(y+3) & 2x(x-2) \\ y+4 & x+2 \end{bmatrix}$$

We have $J(0, -4) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ so $(0, -4)$ is an unstable note (all trajectories near $(0, -4)$ flow away from it).

Next, $J(2, -4) = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$, so $(2, -4)$ is a saddle point.

And $J(-2, -3) = \begin{bmatrix} 0 & 16 \\ 1 & 0 \end{bmatrix}$, which has eigenvalues ± 4, so this is also a saddle point.