9.2 Linear differential equations with constant coefficients

The standard form of a second order differential equation, with y as the independent variable, is:

$$y" + py' + qy = F(t),$$

where $y"$ is by itself, p is a coefficient of y', and $F(t)$ can $= 0$ or $\neq 0$

To solve these constant coefficient equations, we must first solve the homogeneous solution to the equation, and then if it is non-homogeneous, solve that way.

The general solution to any second order differential equation is:

$$y = y_c + y_p,$$

where y_c is the homogeneous solution, or complimentary solution, and y_p is the non-homogeneous solutions, or the particular solution.

Solving Homogeneous second order DEs
Solving equations of the form $y" + by' + cy = 0$ is rather straightforward. Essentially, we know that the solutions will assume the form $y = e^{kt}$, since e^{kt} is an eigenfunction of the differential operator.

Don't worry about this too much, but essentially, the function e^{kt} has the same properties of eigenvectors when differentiating them, since multiplying them by a scalar is the same as differentiating them. With this knowledge, we are able to solve homogeneous second order equations by converting them to their **characteristic polynomial**, which is just changing $y"+by'+cy = 0 \rightarrow r^2+br+c = 0$.

We solve this polynomial to determine the roots, and based on the format of the roots, we can form the general solution to the homogeneous differential equation.

Root style	**General solution**
real, distinct (r_1, r_2)	$y_c = c_1e^{r_1t} + c_2e^{r_2t}$
complex, distinct $(r = a \pm bi)$	$y_c = c_1e^{at}cos(bt) + c_2e^{at}sin(bt)$
real, repeated $(r = r_1 = r_2)$	$y_c = c_1e^{rt} + c_2te^{rt}$

As we can see, there are always two solutions to a second order differential equation, which are linearly independent from each other, even when there is a repeated root to the characteristic polynomial. Each of the solutions are linearly combined, so there is a constant c_1, c_2 multiplied by each of the solutions and added together, similar to our linear combinations from earlier. These constants are essentially constants of integration, and their values depend on the initial values of the equation.

We can solve for the constants in an **initial value problem**, where we are given two initial values, and then are solving a system of equations.

Solving Non-homogeneous second order DEs
If we have a differential equation $y" + by' + cy = F(t)$, where $F(t) \neq 0$, we have to solve for the particular solution, y_p, **after** we solve for the homogeneous solution, y_c. To do this, we use the **Method of Undetermined Coefficients**, which essentially states that we know that our particular solution will take the form of $F(t)$.
For example, if $F(t) = 8e^{5t}$, we know y_p will be of the form Ae^{5t}, but we don't know the coefficient, A. The main point of this method is to make a "guess" of what the y_p looks like, with an "undetermined coefficient", then plug that guess into the differential equation, and then solve for that coefficient, which results in y_p.

This table represents what potential $F(t)$'s can look like and then what their "guess" should be:

F(t)	**Guess,** α
Me^{at}	Ae^{at}
Mt^2	$At^2 + Bt + C$
$Mcos(at)$	$Acos(at) + Bsin(at)$
$Msin(at)$	$Acos(at) + Bsin(at)$

Further, an important note about guesses — the particular solution to a non-homogeneous differential equation must be linearly independent from the homogeneous solution. You must solve for the homogeneous solution before making your guess, because the homogeneous solution will influence your guess. If the $F(t)$ has the same term as the homogeneous solution, you must make sure that your guess does not replicate the term in the homogeneous solution, and this is done by multiplying the guess you would normally make by t. This is similar to how the solutions to the case with repeated roots has the same solutions but one is multiplied by t. The case of y_c and $F(t)$ overlap is relatively rare, but if the case arises and is not dealt with correctly, the solution will be incorrect.

Let's do an example:

Find the general solution to the differential equation $y" + 5y' + 4y = e^{3t}$ with initial values $y(0) = 3, y'(0) = 1$.

Step 1: Solve for the homogeneous solution, y_c

$$\text{Solving } y" + 5y' + 4y = 0$$
$$y" + 5y' + 4y = 0 \rightarrow r^2 + 5r + 4 = 0$$
$$(r+4)(r+1) = 0 \rightarrow r = -4, -1$$
$$y_c = c_1 e^{-4t} + c_2 e^{-t}$$

Step 2: Solve for particular solution, y_p

Solving using the method of undetermined coefficients

Since $F(t) = e^{3t}$, our guess is $\alpha = Ae^{3t}$, we don't need to multiply by t since e^{3t} is not in y_c

$$\alpha = Ae^{3t},\ \alpha' = 3Ae^{3t},\ \alpha" = 9Ae^{3t}$$

$$\text{Plugging in: } 9Ae^{3t} + 5(3Ae^{3t}) + 4(3Ae^{3t}) = e^{3t}$$

$$\text{Distribute and combine: } 28Ae^{3t} = e^{3t}$$
$$28A = 1 \to A = \tfrac{1}{28} \to y_p = \tfrac{1}{28}e^{3t}$$

Step 3: Form general solution

$$y = y_c + y_p$$
$$y = c_1e^{-4t} + c_2e^{-t} + \tfrac{1}{28}e^{3t}$$

Step 4: Solve for constants with initial values

$$y(0) = 3 \to 3 = c_1e^{-4(0)} + c_2e^{-0} + \tfrac{1}{28}e^{3(0)}$$
$$3 = c_1 + c_2 + \tfrac{1}{28} \to \tfrac{83}{28} = c_1 + c_2$$
$$y'(0) = 1 \to 1 = -4c_1e^{-4(0)} + -c_2e^{-0} + \tfrac{3}{28}e^{3(0)}$$
$$1 = -4c_1 - c_2 + \tfrac{3}{28} \to \tfrac{25}{28} = -4c_1 - c_2$$

We can solve this system:

$$\left[\begin{array}{cc|c} 1 & 1 & \frac{83}{28} \\ -4 & -1 & \frac{25}{28} \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -\frac{9}{7} \\ 0 & 1 & \frac{17}{4} \end{array}\right]$$

This tells us $c_1 = -\frac{9}{7}$ and $c_2 = \frac{17}{4}$

The initial value problem solution is thus:

$$y = -\tfrac{9}{7}e^{-4t} + \tfrac{17}{4}e^{-t} + \tfrac{1}{28}e^{3t}$$

We now know how to solve any constant coefficient second order differential equation with familiar non-homogeneous terms. Further studies into differential equations will include learning new methods for solving more complex problems, where there are non-constant coefficient terms, and very unfamiliar non-homogeneous terms.