5 Change of basis, rank/nullity theorem, linear transformations

5.1 Consequences of the rank/nullity theorem

Matrix Inverse Theorem
We have covered a lot of fundamental concepts about matrices, and in this section, we are going to synthesize a lot of information into the **matrix inverse theorem**, which is a really important result in linear algebra. **This only applies to square matrices**.

First, we learned about how we can represent systems of linear equations as matrices and thus find the RREF of a matrix in week 2. In week 4 we also learned about the idea of **rank**, where the rank is the number of non-zero rows in the RREF of a matrix.

For any $n \times n$ matrix, A, if rank $= n$, the RREF of the matrix is the identity matrix.
We know that if $RREF(A) = I$, A^{-1} exists / A is invertible.
Knowing that this means that $Rank(A) = n$, and our knowledge of the rank nullity theorem, it follows $Nullity(A) = 0$.

The determinant is an important property of all square matrices, and it comes into play here. We know that the for a matrix with full rank, it will have a non-zero determinant. We can expand a bit further here and assert that $Det(A) \neq 0 \Rightarrow RREF(A) = I$ and its opposite is true, $Det(A) = 0 \Rightarrow RREF(A) \neq I$.

Moving along, we also discussed in week 2 the types of solutions to the expression $Ax = b$ for any A. Now, for square A matrices, we know that if A^{-1} exists, then $Ax = b$ has unique solutions for all b. Further, the equation, $Ax = 0$, which represents the nullspace, only has the **trivial solution**, which means it is the empty space (only when all values are 0).

Further, when $Rank(A) = n$, this tells us that we have n linearly independent rows and columns (# nonzero rows in RREF = # nonzero columns in RREF). From our discussion of $\mathbb{R}^n$ spaces & dimension, if we have n linearly independent $n \times 1$ vectors, we can span $\mathbb{R}^n$, so this applies to our rows and columns.

Putting this all together:

Matrix Inverse Theorem
($n \times n$ matrices)
$\Leftrightarrow A^{-1}$ exists
$\Leftrightarrow RREF(A) = I \Leftrightarrow Det(A) \neq 0$
$\Leftrightarrow Rank(A) = n \Leftrightarrow Nullity(A) = 0$
$\Leftrightarrow$ columns/rows are linearly independent
$\Leftrightarrow Ax = b$ has unique solutions $\Leftrightarrow Ax = 0$ has only the trivial solution
$\Leftrightarrow ColSp(A) = \mathbb{R}^n \Leftrightarrow RowSp(A) = \mathbb{R}^n \Leftrightarrow NullSp(A) = 0$