4.2 Change of basis, the rank/nullity theorem

Change of basis

Now that we have a great idea of bases, we can discuss the concept of change of basis.
Essentially, we know that for any vector space, the dimension and span is conserved. We represent this with a basis set of vectors, polynomials, or matrices. However, we are able to use infinitely many bases for a given vector space. To discuss this further, we need to break down exactly how a vector is a linear combination of its basis vectors.

When we have any given element, whether it is a vector, polynomial, or matrix, it exists in the context of its basis. In the previous section, we described the **standard bases** of common vector spaces.

The standard basis of $\mathbb{R}^3$ is: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$, which tells us that any 3×1 vector can be written as a linear combination of these three elements. In fact, any 3×1 vector **is** written as a linear combination of these three elements.

For example:

$$\begin{bmatrix}4\\-3\\6\end{bmatrix} = 4\begin{bmatrix}1\\0\\0\end{bmatrix} + -3\begin{bmatrix}0\\1\\0\end{bmatrix} + 6\begin{bmatrix}0\\0\\1\end{bmatrix}$$

While this expression seems very obvious, it is important to recognize for generalizability that each vector in a vector space is actually represented as a linear combination of its basis elements. In the above example, the vector represents the coefficients of the basis elements, in this case the standard basis, needed to represent the vector.

$$\begin{bmatrix}{\color{red}4}\\{\color{blue}-3}\\{\color{purple}6}\end{bmatrix} = {\color{red}4}\begin{bmatrix}1\\0\\0\end{bmatrix} + {\color{blue}-3}\begin{bmatrix}0\\1\\0\end{bmatrix} + {\color{purple}6}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Because this is true, we could also write the vector like this:

$$\begin{bmatrix}4\\-3\\6\end{bmatrix}_S$$

The S tells you that the vector is being expressed as a linear combination of the standard basis, S, of $\mathbb{R}^3$. However, there are infinitely many bases for $\mathbb{R}^3$, since all we need to comprise a basis for $\mathbb{R}^3$ are 3, 3×1 linearly independent vectors. Another basis for $\mathbb{R}^3$ can be called B_1, where:

$$B = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$$

We now need to solve for the coefficients used to generate $\begin{bmatrix}4\\-3\\6\end{bmatrix}$ from these basis vectors, solving:

$$\begin{bmatrix}4\\-3\\6\end{bmatrix}_S = c_1\begin{bmatrix}1\\2\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ then } \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_{B_1}$$

It is rather obvious that this is simply a system of linear equations, which we know how to solve, so we see this can also be represented as:

$$\begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}_S = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{B_1} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}_S = \begin{bmatrix} -2.5 \\ 2 \\ 6.5 \end{bmatrix}_{B_1} = -2.5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 6.5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The matrix with the basis elements column-wise is called the **change of basis matrix**. It allows us to convert from one basis to another. This matrix can be denoted $P_{B_1 \to S}$ since it maps vectors in B_1 to vectors in S, as we can see by the first expression above. In the second expression, though, after we did some rearranging, the inverse of $P_{B_1 \to S}$ maps vectors in S to vectors in B_1. This is an important discovery because we observe for any change of basis matrix, its inverse reverses the changing of basis:

$$P_{S \to B} = (P_{B \to S})^{-1} \text{ and } (P_{B \to S})^{-1} = P_{S \to B}$$

It is important to note that for any basis, when put column-wise into a matrix to generate a change of basis matrix, it is **always** denoted, for any generic basis, B as $P_{B \to S}$, as in mapping from the original basis to the respective standard basis. This even is true for the standard basis itself, where

$$P_{S \to S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Back to our example above, since our basis B_1 was for $\mathbb{R}^3$, it created a square change of basis matrix, $P_{B_1 \to S}$, which allowed us to find the inverse of the matrix, $P_{S \to B_1}$.
While this is easier, you may encounter bases for spaces that are different sizes, whether it is a subspace of $\mathbb{R}^3$, or something similar, we could have also solved the above example like this:

$$\begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}_S = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{B_1} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{B_1} = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & -3 \\ 1 & 1 & 1 & 6 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2.5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6.5 \end{array}\right] \therefore \begin{bmatrix} -2.5 \\ 2 \\ 6.5 \end{bmatrix}_{B_1}$$

To summarize the idea of change of basis:

- There are infinitely many bases for any given vector space, and every basis for a given space has the same dimension and span
- Every vector in a vector space is represented by the coefficients used to generate the vector using its basis elements, traditionally we use the standard basis
- When changing between bases, we can multiply a vector that is currently with respect to one basis by a **change of basis (COB) matrix** to change it to another basis: $P_{S \to B} v_S = v_B$
- Taking the inverse of a COB matrix changes the direction of the change: $P_{S \to B} = (P_{B \to S})^{-1}$
- We can "squish" together COB matrices to move between multiple bases: $P_{S \to B} P_{B \to C} = P_{S \to C}$

With this wealth of knowledge of bases and spaces, let's introduce three very important spaces that we will be using for the rest of this course.

Row space

The row space of a matrix is the set of vectors that can be generated from the rows in the RREF of a matrix. We are able to find the basis for the row space by performing Gaussian Elimination, finding the non-zero rows in the RREF, and that is the basis for the row space.

Further, the dimension of the row space is called the **rank**, and this is the number of non-zero rows in the RREF. This is also the number of pivot variables. The rank is an extremely important value in a matrix.

Column space

The column space of a matrix is the set of vectors that can be generated from the columns in the RREF of a matrix. We are able to find the basis for the column space by performing Gaussian Elimination, finding the non-zero columns in the RREF, and then the basis for the column space are the associated columns in the **original matrix** that are in the same columns as those in the RREF. The dimension of the column space is going to be the same as the dimension of the row space, because for each non-zero row there will be a non-zero column.

This is slightly different than the row space because when we perform row operations, the span of the rows is preserved since we are performing linear operations based on rows. Therefore, the span of the original rows in a matrix is the same as the span of the RREF rows of a matrix. However, when we do row operations, the span of the columns do change, so the span of the original columns in a matrix is **not** the same as the span of the RREF columns of a matrix.

Therefore, in order to represent the column space of a matrix, we must use the original columns in a matrix, but to determine which columns those are, we find the nonzero columns in the RREF, which correspond to the pivot variables of the matrix.

Here is an example of solving the basis for the row space, column space, and rank of a matrix :

$$\begin{bmatrix} 2 & 2 & 0 \\ 3 & 1 & 1 \\ 4 & 4 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis for row space: } \left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\text{Basis for column space: } \left\{ \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$$

As we can see, the basis for the row space corresponds to the three nonzero rows in the RREF. The rank of the matrix is 3 since there are 3 nonzero rows in the RREF. The column space also has a dimension of 3, since there are 3 nonzero columns in the RREF.

The basis, however, is comprised of the original columns of the matrix, not those from the RREF. Since there is a pivot variable in column 1, 2, and 3, the associated columns from the original matrix are columns 1, 2, and 3.

Nullspace

One last fundamental space of a matrix to cover is the nullspace. This is also known as the kernel. The nullspace comprises all solutions for a given matrix A to the equation $Ax = 0$. The nullspace is an extremely important space for a matrix because it tells us a lot about a matrix and its properties. Solving for the basis for the nullspace is the same as solving for solutions to $Ax = 0$, which is a special case of solving $Ax = b$, just when the system is homogeneous.

The dimension of the basis for the nullspace is called the **nullity**. We will discuss how this fits in with the rank of the matrix in the next section.

Here is an example:
Find the nullity and basis for the nullspace of the following matrix, A:

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & -3 & 1 \\ 1 & -2 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the RREF of $A|0$, we have two nonzero rows with two pivot variables, x_1, x_2, and one free variable x_3. We will solve as we did with $Ax = b$ systems by extracting the equations from the matrix and writing the pivot variables in terms of the free variables.

$$x_1 + x_3 = 0 \rightarrow x_1 = -x_3$$
$$x_2 = 0$$
$$x_3 = x_3$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3, \text{ therefore NullSp(A)} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

From the example above, we see that we had two pivot variables and one free variable, x_3, where the solution to $Ax = 0$ is written in terms in x_3, so we only have one vector in the basis for the NullSp(A), so the nullity = 1. This leads us to the Rank/Nullity Theorem.

Rank/Nullity Theorem
The Rank-Nullity Theorem is an extremely important result in the world of linear algebra because it connects the row space of matrices with the nullspaces of the matrices. In short, for any $m \times n$ matrix, A, where n is the number of columns:

$$\mathbf{Rank(A) + Nullity(A) = n}$$

This result should make sense from topics 8 and 9 before this. The number of pivot variables in the RREF of a matrix corresponds to the rank of that matrix, and then the number of free variables in the RREF of a matrix corresponds to the nullity of that matrix.

Further, # pivot variables + # free variables = # columns. This is the Rank-Nullity theorem in disguise.

With this information, if we know the rank of our matrix, we know the nullity of the matrix. This is true for square and non-square matrices. However, for square matrices, this tells us that also # pivot variables + # free variables = # columns = # rows.

The row space of a matrix is orthogonal to the nullspace of a matrix. This makes sense because the nullspace, by definition, is $Ax = 0$, so for any row in the matrix, in RowSp(A), A, it is dotted with a vector in x to get 0. All of the vectors in x are in the NullSp(A).