4 Linear independence, basis and dimension

4.1 Linear independence, bases and dimension

Linear independence

Two vectors (or functions) are **linearly independent** when there is no way to multiply one vector by a scalar number $\in \mathbb{R}$ to get the other vector. In other words, one vector is not a **scalar multiple** of the other vector.

Three vectors (or functions) are **linearly independent** when there is no way to combine two of the vectors to get the other vector, and vice versa. Let our three vectors be denoted v_1, v_2, v_3, the combination of them can be written as $c_1v_1 + c_2v_2 = c_3v_3$, and $c_1, c_2, c_3 \in \mathbb{R}$. This can also be rewritten as $c_1v_1 + c_2v_2 - c_3v_3 = 0$

If there is some combination where this is true, and $c_1 \neq 0$, and/or $c_2 \neq 0$, and/or $c_3 \neq 0$, there there is a combination of them, and they are **linearly dependent**. If the only way for this combination to work is when $c_1 = c_2 = c_3 = 0$, then they are **linearly independent**.

To determine if they are linearly dependent or linearly independent, we write the vectors column-wise in a matrix and then perfect Gaussian Elimination. If the RREF of the matrix is the identity matrix, they are **linearly independent**, and if the RREF is **not** the identity matrix, then they are **linearly dependent**. Here are examples of both:

Linearly Independent Example

$$\begin{bmatrix}-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}-2\\3\\-2\end{bmatrix}$$

$$\begin{bmatrix}-3 & 1 & -2\\2 & -1 & 3\\-3 & 3 & -2\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Therefore the vectors are linearly independent. The only linear combination of these vectors that works is when $c_1 = c_2 = c_3 = 0$.

$$0 \times \begin{bmatrix}-3\\2\\-3\end{bmatrix} + 0 \times \begin{bmatrix}1\\-1\\3\end{bmatrix} = 0 \times \begin{bmatrix}-2\\3\\-2\end{bmatrix}$$

Linearly Dependent Example

$$\begin{bmatrix}2\\-4\\5\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}, \begin{bmatrix}-1\\-6\\9\end{bmatrix}$$

$$\begin{bmatrix}2 & 3 & -1\\-4 & 2 & -6\\5 & -4 & 9\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

Therefore the vectors are linearly dependent. Further, there are infinitely many combinations of these vectors. From the RREF, $c_1 - c_3 = 0$ and $c_2 + c_3 = 0$. A combination that would work is $c_1 = 1, c_2 = -1, c_3 = 1$.

$$1 \times \begin{bmatrix}2\\-4\\5\end{bmatrix} + -1 \times \begin{bmatrix}3\\2\\-4\end{bmatrix} = 1 \times \begin{bmatrix}-1\\-6\\9\end{bmatrix}$$

The concept of the linear independence of vectors applies to all vectors and functions, too. This method can be used to test the linear independence of any size or number of vectors.

For example, to determine the linear independence of 2 6×1 vectors, you would put them in column-wise into a 6×2 matrix, perform Gaussian Elimination, and if the first two rows become the identity, then they are linearly independent. If they are not, they are linearly dependent. This is because when we do this, we are solving for the constants that combine the columns. If it is the identity, each constant is set $= 0$ and they are linearly independent. If not, that information tells us how the constants are related.

Further, using our knowledge of the Matrix Inverse Theorem, since we know that $RREF(A) = I \Leftrightarrow Det(A) \neq 0$, if we are determining the linear independence of 2 2×1's, 3 3×1's, ..., n $n \times 1$'s, then we can either do Gaussian Elimination, or see if $Det(A) \neq 0$.

$Det(A) \neq 0 \Leftrightarrow RREF(A) = I \Leftrightarrow$ columns are linearly independent

$Det(A) = 0 \Leftrightarrow RREF(A) \neq I \Leftrightarrow$ columns are linearly dependent

Span & Basis

Now that we have covered linear independence, we can talk about the concepts of span and basis. When we have a set of vectors, we can linearly combine them in infinitely many ways to get infinitely many vectors. The "space" of vectors that these vectors can make with these combinations is called the **span** of the vectors. We can use any vectors to span any space. Here is an example:

$$Span\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right) = \mathbb{R}^3$$

This tells us that we can "generate" or make any vector in $\mathbb{R}^3$ (think any vector in the x, y, z space) from those vectors. Those vectors **span** $\mathbb{R}^3$. Also, the **span** of those vectors is $\mathbb{R}^3$. A.k.a. **span** can be a noun or a verb.

When a vector is in the span of vectors, it is a linear combination of those vectors. Using the example from above:

$$\begin{bmatrix}5\\9\\-4\end{bmatrix} \in Span\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right), \begin{bmatrix}5\\9\\-4\end{bmatrix} = 5 \times \begin{bmatrix}1\\0\\0\end{bmatrix} + 9 \times \begin{bmatrix}0\\1\\0\end{bmatrix} + -4 \times \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Determining if a vector is a linear combination of other vectors is the same as determining of a vector is in the span of those vectors.

A basis is a specific set of vectors that are the vectors needed to span a certain space. Basis vectors must be **minimally spanning**, so if you remove one of the vectors in the set it no longer spans that space. Basis vectors must also be linearly independent.

Standard bases are the normal bases used to represent common spaces, most commonly spaces like $\mathbb{R}^2, \mathbb{R}^3, P_3, \mathbb{R}^{2x2}$.

$\mathbb{R}^2$ is the space of all vectors in the x, y plane, and its standard basis is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathbb{R}^3$ is the space of all vectors in the x, y, z plane, and its standard basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

P_3 is the space of all polynomials (functions) up to the degree of 3. its standard basis is $\{1, x, x^2, x^3\}$

$\mathbb{R}^{2x2}$ is the space of all 2×2 matrices, and its standard basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Each of these are the normal basis for each of these spaces. If you remove one of these vectors/polynomials/matrices, you can no longer generate/span the same space. If I remove $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ from the basis for $\mathbb{R}^3$, then I can no longer make any vector with a x coordinate. Further, there are infinitely many ways to represent a basis for $\mathbb{R}^3$ because all you need are 3 3×1 linearly independent vectors.
Other important spaces are the **nullspace** and **column space**. The nullspace is the set of solutions, x, to $Ax = 0$. The column space is the space spanned by the columns of A.

This tells us that for every space, there is always the same amount of vectors in the basis. The **dimension** is the number of elements in a basis. The dimension is always the same for a specific space.
$dim(\mathbb{R}^3) = 3$
$dim(\mathbb{R}^2) = 2$
$dim(P_3) = 4$
$dim(\mathbb{R}^{2x2}) = 4$
$dim(Nullspace) = nullity$
$dim(Columnspace) = image$

To generalize these, $dim(\mathbb{R}^n) = n$, $dim(P_d) = d + 1$, $dim(\mathbb{R}^{nxn}) = n^2$

Overall, the concepts of linear independence, spans, bases, and dimensions are really important to linear algebra. Bases are used to represent the span of a space, and we use those for many applications.