3.2 Vector spaces and subspaces

Pivoting a bit from matrices, we are entering the more abstract concepts in linear algebra. To begin this discussion, we have to introduce the idea of vector spaces (and consequently subspaces).

Vector spaces

By definition, a vector space is a collection of vectors or polynomials (in this class) that have some specific condition describing them, and that when you add two (or more) of these elements together, or multiply an element by a scalar (a real number, in this course), you will still have a vector or polynomial, with that specific condition.

To start simply, one of the most commonly referred to vector spaces in this course is $\mathbb{R}^3$, which consists of all 3×1 vectors with real number components, $\in \mathbb{R}$. This is a very familiar collection of vectors, since they are essentially any position in our 3D world. Now, imagine you had a position, any position, and added it with another position, any position. You would still have a position in the 3D world. There is no combination of $\mathbb{R}^3$ vectors that can add together to not belong to $\mathbb{R}^3$. Similarly, there is no **real** number (scalar) you could multiply a 3×1 vector by and not have a 3×1 vector. You cannot *escape* the 3×1 vector realm, you cannot escape that space.

To formalize this definition a bit, a vector space must:

- observe closure under addition/subtraction
- observe closure under scalar multiplication
- contain the "zero vector"

The "zero vector" refers to the analogous item in a space that is equal to 0. For example, in $\mathbb{R}^3$, it would be the 3×1 zero vector, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. For any polynomial space, it would be the function that equals zero, $f(x) = 0$. Essentially, you just have to ensure that whatever property defines the vector space can allow for a component to be 0.

Some examples of spaces that would not contain the zero vector are polynomials in P_3 such that $f(0) = 7$, or all 2×1 vectors where $b_{2,1} = 5$.

The zero vector property is mostly used to **disprove** a vector space, rather than to prove its existence.

Vector subspaces

A vector subspace is a vector space, but also a subset. **Sub** in subspace comes from **Sub**set and **space** in subspace comes from vector **space**. The only major distinction is that these subspaces belong to a larger vector space, traditionally $\mathbb{R}^n$ or other spaces, and have a specific property that is only true for a subset of that space, as mentioned above, such as $f(8) = 0$ in P_3. Since these subspaces are vector spaces, they must have the same three properties as described above, as well as simply being a subset of that space, which is typically obvious.

Proving vector spaces

In this course, you will be given a space and asked if it is a valid vector (sub)space. Rule of thumb, you need to prove **arbitrarily** but can disprove **specifically**.

This means that if you are asserting that a space *is* a vector space, you must prove that for any arbitrary elements in that space, f, g, it obeys the above properties. It must be **generalized**.

However, if you are asserting that a space *is not* a vector space, you can simply provide a contradiction or counterexample to disprove it. This is a general idea in proofs but comes into play in this section.

We need to determine, given two arbitrary elements in S, if we add two of them together, we will still $\in S$, if we multiply by one scalar, we will still be $\in S$, and that there is an analogous 0 value.

Let's do some examples:

Let the subset, S of $\mathbb{R}^4$ be described as:

$S = \{(x_1, x_2, x_3, x_4) \mid x_1 = x_3, x_2 = x_4\}$

Prove or disprove this as a vector subspace of $\mathbb{R}^4$.

<u>Closure under addition</u>

Let $p, q \in S$ (assert there are two arbitrary elements $\in S$ that are in the space)

Therefore $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_1 \\ p_2 \end{bmatrix}$ and $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_1 \\ q_2 \end{bmatrix}$. (take the property and generalize it for p, q)

We see $\begin{bmatrix} p + q \end{bmatrix} = \begin{bmatrix} p_1 + q_1 \\ p_2 + p_2 \\ p_1 + q_1 \\ p_2 + q_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_1 \\ q_2 \end{bmatrix} = p + q$. (vector summation = individual summation)

$\therefore S$ is closed under addition.

<u>Closure under scalar multiplication</u>

Let $p \in S$ and $c \in \mathbb{R}$

Therefore $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_1 \\ p_2 \end{bmatrix}$

We see $\begin{bmatrix} c \times p \end{bmatrix} = \begin{bmatrix} cp_1 \\ cp_2 \\ cp_1 \\ cp_2 \end{bmatrix} = c \times p = \begin{bmatrix} cp_1 \\ cp_2 \\ cp_1 \\ cp_2 \end{bmatrix}$. (vector scaling = individual scaling)

$\therefore S$ is closed under scalar multiplication.

<u>Contains zero vector</u>

The analogous zero vector $\in S$ is the 4×1 zero vector, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

We see that $x_1 = x_3 = 0$ and $x_2 = x_4 = 0$. $\therefore S$ is closed under scalar multiplication.

$\therefore S$ is a valid subspace of $\mathbb{R}^4$ since it is a valid vector space and obviously belongs to $\mathbb{R}^4$.

The previous proof was very rigorous, you do not need to be this thorough — just be sure to prove all three axioms arbitrarily.

Let the subset, S of $\mathbb{R}^3$ be described as:

$$S = \left\{ v \in \mathbb{R}^3 : v = \begin{bmatrix} 1+t \\ t \\ 2t \end{bmatrix}, t \in \mathbb{R} \right\}$$

Prove or disprove this a vector subspace of $\mathbb{R}^3$

<u>Closure under addition</u>

Let $p, q, \in S$

Therefore $p = \begin{bmatrix} 1+x \\ x \\ 2x \end{bmatrix}$ and $q = \begin{bmatrix} 1+y \\ y \\ 2y \end{bmatrix}$, where $x, y \in \mathbb{R}$ and may or may not be unique.

We see $\begin{bmatrix} p+q \end{bmatrix} = \begin{bmatrix} 1+x+1+y \\ x+y \\ 2x+2y \end{bmatrix} = \begin{bmatrix} 2+(x+y) \\ x+y \\ 2(x+y) \end{bmatrix}$, where regardless of $x = y$ or $x \neq y$, does not follow the form $\begin{bmatrix} 1+t \\ t \\ 2t \end{bmatrix}$ due to the first row. $\therefore S$ is **not** closed under addition. We are done.

This space also fails closure under scalar multiplication, and does not contain the zero vector. You can prove that this fails under any of these axioms and you can be done.

Further, closure under addition failed above arbitrarily, but you could also disprove by counterexample:

Let $p = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $(t = 1)$ and $p = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}$ $(t = 3)$.

We see $\begin{bmatrix} p+q \end{bmatrix} = \begin{bmatrix} 1+(1+3) \\ 1+3 \\ 2(1)+2(3) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 8 \end{bmatrix} \neq p+q = \begin{bmatrix} 2+4 \\ 1+4 \\ 2+8 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 10 \end{bmatrix}$.

This proof for **disproving** vector spaces is equally valid as the first proof. It is truly ideal when solving these problems if you can immediately identify a counterexample, demonstrate that one of the axioms fail, and then conclude. If you cannot generate a counterexample, then you should attempt to prove the vector spaces arbitrarily like the previous example.