3 Determinants, vector spaces and subspaces

3.1 Determinants

The determinant of a matrix is one of the more important properties of a matrix. Whether it is $= 0$ or $\neq 0$ tells you so much about the matrix itself and the concepts it represents. The determinant is a property unique to square matrices, so if a matrix does not have the same number of rows as columns then you cannot calculate the determinant of that matrix.

The determinant of a matrix is a scalar value that is calculated from the entries of the matrix, and is denoted as $det(A)$ for matrix A.

This is how the matrix is calculated for a 1×1,2×2, and 3×3, which are the most relevant for MATH 2400. Analytically, you can see how these methods apply for larger matrices.

Determinant of square matrices

Det(1×1)

$$A = \begin{bmatrix} a \end{bmatrix},\ det(A) = \begin{vmatrix} a \end{vmatrix} = a$$

Det(2×2)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Det(3×3)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

The determinant is also relevant to the matrix geometrically, where the magnitude and sign of the determinant tells you what happens to vectors when multiplied by that matrix, but this is less relevant now.

An important property of matrices, further, is that $Det(AB) = Det(A) \times Det(B)$. This tells us that the product of two matrices has the same determinant of the product of the individual matrices.

This is also interesting because while the order of matrix multiplication matters, which we discussed earlier, so $A \times B \neq B \times A$, we see that $Det(AB) = Det(BA) = Det(A) \times Det(B) = Det(B) \times Det(A)$.

Effects of row operations on determinants

It is also important to talk about what each row operation does to the determinant.

Type 1: Switching two rows	Type 2: Scaling a row	Type 3: Adding a scaled row to another row
Let the original matrix be A and the matrix with the swapped rows be A'. $Det(A') = -Det(A)$	Let the original matrix be A and the matrix with any of its rows multiplied by scalar c be A'. $Det(A') = c \times Det(A)$	Let the original matrix be A and the matrix with any of its rows multiplied by scalar c and added to another row be A'. $Det(A') = Det(A)$

As we can see, Type 1 row operations negate the determinant, Type 2 row operations multiply the determinant by the scalar, and Type 3 operations have no effect on the determinant.

Determinant properties summary

$Det(I) = 1$ where I is any sized *identity* matrix
$Det(AB) = Det(A) \times Det(B)$
$Det(c \times A) = c^n \times Det(A)$ where n is the size of the matrix
If there is a row/column of all 0's, $Det(A) = 0$
Determinant of upper/lower triangular or diagonal matrices is the product of the diagonal entries
$Det(A^T) = Det(A)$
$Det(A^{-1}) = Det(A)^{-1}$