2.2 Inverse of a square matrix

You can only calculate the inverse of a **square matrix**. The inverse, denoted for any matrix A as A^-1, is used as the reversal of matrix multiplication and allows us to solve for matrices by eliminating matrices since for every matrix A, $AA^{-1} = A^{-1}A = I$.

Therefore, if we have the expression, $Ax = b$, for any system of equations and we want to solve for x, which is our goal with systems of equations, we need to isolate x by multiplying on the left by A^{-1}, $A^{-1}Ax = A^{-1}b$, which becomes $x = A^{-1}b$.

You are always able to find the transpose of a matrix, but **not all square matrices have inverses**. If the RREF of the matrix is the identity matrix, then it has an inverse. If it is not, then it **does not have an inverse**.

Calculating the inverse of (square) matrices

In MATH 2400, you could be asked to solve for the inverse of 2×2, 3×3, or even occasionally 4×4 matrices. Let's talk about how to solve each of them.

Inverse of 2×2 *matrices*
Luckily, this is easy and we can follow a formula.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ the inverse is: } A^{-1} = \tfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of 3×3 *or* 4×4 *matrices*
Unfortunately, solving for any square matrices that are not 2×2 matrices requires a lot more work. To solve for the inverse of these matrices, you must augment the matrix with the corresponding identity matrix, and then perform Gaussian Elimination on the original matrix to find the RREF of the matrix. The row operations you do to the original matrix you must also do to the identity matrix, and once you complete the row reduction:

If the RREF is the identity matrix, then the resultant reduced matrix on the right is the inverse.

If the RREF is not the identity matrix, then the matrix **does not** have an inverse, and it is called **singular**, or **non-invertible**.

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textit{Gaussian Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

Let's do an example of the inverse of a 3×3 matrix. Again, this method works for all square matrices, and you would do the same method for 4×4, or even 5×5 matrices. If you are asked to determine the **invertibility** of a matrix, whether or not it has an inverse, then you must perform these operations and determine if the left hand matrix becomes the identity matrix or not.

Find A^{-1}, if it exists, where $A = \begin{bmatrix} 5 & 10 & 5 \\ 3 & 12 & -9 \\ 4 & -4 & 12 \end{bmatrix}$ (same as earlier RREF example)

Step 1: Augment with the identity matrix

$$\left[\begin{array}{ccc|ccc} 5 & 10 & 5 & 1 & 0 & 0 \\ 3 & 12 & -9 & 0 & 1 & 0 \\ 4 & -4 & 12 & 0 & 0 & 1 \end{array}\right]$$

Step 2: Perform Gaussian Elimination until the original matrix is in RREF

$$\left[\begin{array}{ccc|ccc} 5 & 10 & 5 & 1 & 0 & 0 \\ 3 & 12 & -9 & 0 & 1 & 0 \\ 4 & -4 & 12 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1=\frac{1}{5}R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{5} & 0 & 0 \\ 3 & 12 & -9 & 0 & 1 & 0 \\ 4 & -4 & 12 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2=R2-3R1} \xrightarrow{R3=R3-4R1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{5} & 0 & 0 \\ 0 & 6 & -12 & -\frac{3}{5} & 1 & 0 \\ 0 & -12 & 8 & -\frac{4}{5} & 0 & 1 \end{array}\right] \xrightarrow{R3=R3+2R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{5} & 0 & 0 \\ 0 & 6 & -12 & -\frac{3}{5} & 1 & 0 \\ 0 & 0 & -16 & -2 & 2 & 1 \end{array}\right] \xrightarrow{R3=-\frac{1}{16}R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{5} & 0 & 0 \\ 0 & 6 & -12 & -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \end{array}\right] \xrightarrow{R2=R2+12R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{5} & 0 & 0 \\ 0 & 6 & 0 & \frac{9}{10} & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \end{array}\right] \xrightarrow{R2=\frac{1}{6}R2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{20} & -\frac{1}{12} & -\frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \end{array}\right] \xrightarrow{R1=R1-R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{3}{40} & -\frac{1}{8} & -\frac{1}{16} \\ 0 & 1 & 0 & \frac{3}{20} & -\frac{1}{12} & -\frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \end{array}\right] \xrightarrow{R1=R1-2R2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{40} & -\frac{7}{24} & \frac{5}{16} \\ 0 & 1 & 0 & \frac{3}{20} & -\frac{1}{12} & -\frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \end{array}\right]$$

$$\mathbf{A^{-1}} = \begin{bmatrix} -\frac{9}{40} & -\frac{7}{24} & \frac{5}{16} \\ \frac{3}{20} & -\frac{1}{12} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \end{bmatrix}$$

Properties of Inverses

$$(A^{-1})^{-1} = A \qquad (AB)^{-1} = A^{-1}B^{-1} \qquad (A^T)^{-1} = (A^{-1})^T$$