11.2 Constant coefficient systems

We can have many different conditions for this system. Firstly, the system can be homogeneous or non-homogeneous, so $x' = Ax + f$, if $f \neq 0$, the system is non-homogeneous and is a little more complicated to solve. We will avoid this case for this discussion, but in MATH 2400 we dive into this further. When $f = 0$, the system is homogeneous and we can solve it. Further, the matrix A that is formed can either be diagonalizable or defective. When the system is defective/non-diagonalizable, it is more complicated to solve, and we will also avoid this case for the sake of this discussion.

To solve our homogeneous diagonalizable subcases, we must solve for the eigenvalues and eigenvectors of A. The two potential results of solving for this are that the eigenvalues can be real or complex. Let's discuss both possibilities:

Real eigenvalues for A

When solving these systems, you will have n eigenvalues and n eigenvectors for an $n \times n$ matrix. The general solution to the system of differential equations then is as follows for 2×2 and 3×3 matrices:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} v_{\lambda_1} \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} v_{\lambda_2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} \\ v_{\lambda_1} \\ \\ \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} \\ v_{\lambda_2} \\ \\ \end{bmatrix} + c_3 e^{\lambda_3 t} \begin{bmatrix} \\ v_{\lambda_3} \\ \\ \end{bmatrix}$$

These examples show that the solutions to these systems are very similar to our solutions to second order differential equations, where each solution is linearly combined with each other with constants c_1, c_2, c_3. There is an exponential term with the associated eigenvalue and then multiplied by its associated eigenvector. This makes solving diagonalizable systems with real eigenvalues fairly easy.

Here is an example:

Solve the following $x' = Ax$ system: $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

<u>Step 1: Solve for the eigenvalues and eigenvectors</u>

$$Det(A - \lambda I) = (5 - \lambda)^2 - 9 = 0 = (\lambda - 8)(\lambda - 2) = 0 \rightarrow \lambda = 8, 2$$

$\lambda = 8$

$$A - 8I = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$\text{NullSp}(A - 8I) \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 2$

$$A - 2I = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$\text{NullSp}(A - 2I) \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

<u>Step 2: Form the general solution</u>

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} v_{\lambda_1} \\ \ \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} v_{\lambda_2} \\ \ \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Complex eigenvalues for A

When we have a matrix A that has complex eigenvalues, the process is a little bit more involved, especially since we will be dealing with i's in our eigenvalues and eigenvectors. The general solution to these equations is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{at}(cos(bt) + isin(bt)) \begin{bmatrix} v \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = e^{at}(cos(bt) + isin(bt)) \begin{bmatrix} \ \\ v \\ \ \end{bmatrix}$$

where $\lambda = a \pm bi$ and v is the eigenvector for the positive eigenvalue, $\lambda = a + bi$

The best way to explain this process is with a thorough example:

Solve the following $x' = Ax$ system: $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

Step 1: Find the eigenvalues of A

$$Det(A - \lambda I) = (3 - \lambda)^2 + 16 \rightarrow \lambda = 3 \pm 4i$$

Step 2: Find the eigenvectors of A using the **positive** eigenvalue

$$\lambda = 3 + 4i$$

$$A - (3 + 4i)I = \left[\begin{array}{cc|c} -4i & 4 & 0 \\ -4 & -4i & 0 \end{array}\right]$$

$$\text{NullSp}(A - (3 + 4i)I) = \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}$$

Step 3: Plug into general form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{at}(cos(bt) + isin(bt)) \begin{bmatrix} v \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{3t}(cos(4t) + isin(4t)) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Step 4: Distribute multiplied term into vector

$$\begin{bmatrix} e^{3t}cos(4t) + e^{3t}isin(4t) \\ e^{3t}icos(4t) + e^{3t}i^2sin(4t) \end{bmatrix}$$

$$\text{Recall } i^2 = -1$$
$$\downarrow$$
$$\begin{bmatrix} e^{3t}cos(4t) + e^{3t}isin(4t) \\ e^{3t}icos(4t) - e^{3t}sin(4t) \end{bmatrix}$$

Step 5: Regroup vectors with & without i

$$\begin{bmatrix} e^{3t}cos(4t) \\ -e^{3t}sin(4t) \end{bmatrix} + \begin{bmatrix} e^{3t}isin(4t) \\ e^{3t}icos(4t) \end{bmatrix}$$

Step 6: Rewrite with constants

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} e^{3t}cos(4t) \\ -e^{3t}sin(4t) \end{bmatrix} + c_2 \begin{bmatrix} e^{3t}sin(4t) \\ e^{3t}cos(4t) \end{bmatrix}$$

The i in the second vector gets absorbed into c_2, and this is our general solution to the system of differential equations.

We can also solve initial value problems for systems of first order differential equations, where we are given two initial values and solve for c_1, c_2.