10.2 Cauchy-Euler equations

(we will skip the planetary motion applications as described in Canvas)
A specific type of non-constant second order differential equation is the Cauchy-Euler equation, which takes the form (in this course):

$$Ax^2y" + Bxy' + Cy = F(t)$$
$$A, B, C \in \mathbf{R},\ F(t) = 0 \text{ or } \neq 0$$

We solve these equations similarly to constant coefficient systems, where we know what type of solution the equation should have, plug it in, and solve. For Cauchy-Euler equations, we know our solutions will take the form $y = x^p$. We plug this form into the equation, solve for p, and based on the values of p we can find our general solution:

p **style**	**General solution**
real, distinct (p_1, p_2)	$y_c = c_1x^{p_1} + c_2x^{p_2}$
complex, distinct $(p = a \pm bi)$	$y_c = c_1x^a cos(b\ln x) + c_2x^a sin(b\ln x)$
real, repeated $(p = p_1 = p_2)$	$y_c = c_1x^p + c_2x^p \ln x$

Let's do an example:

Solve for the general solution to the Cauchy-Euler equation $x^2y" - 3xy' + 3y = 0$

Step 1: Plug in $y = x^p$

$$y = x^p,\ y' = px^{p-1},\ y" = p(p-1)x^{p-2}$$
$$x^2(p(p-1)x^{p-2}) - 3x(px^{p-1}) + 3(x^p) = 0$$

Step 2: Solve for p values

Rearranging terms, and also recalling $x^{a-b} = x^a x^{-b}$

$$p(p-1)x^2x^{-2}x^p - 3pxx^{-1}x^p + 3x^p = 0$$

Cancel out terms: $p(p-1)x^p - 3px^p + 3x^p = 0$

Pull out x^p: $x^p(p(p-1) - 3p + 3) = 0$

Solve for roots: $p^2 - p - 3p + 3 = p^2 - 4p + 3 = (p-3)(p-1) = 0 \rightarrow p = 1, 3$

Step 3: Form general solution from chart

Using the table, we have two real distinct p values, so our general solution is:

$$y = c_1x^1 + c_2x^3$$

Non-homogeneous Cauchy-Euler equations & initial value problems are solved in the same manner as constant-coefficient differential equations. To solve for y_p, make a guess based off of $F(t)$ with a general coefficient and then solve for it in the equation. With initial values, use those to solve for c_1 and c_2.

$\mathbf{F(x)}$	**Guess,** α
Me^{ax}	Ae^{ax}
Mx^2	$Ax^2 + Bx + C$
$Mln(x)$	$Aln(x)$
$Msin(ln(x))$	$Axsin(ln(x)) + Bxcos(ln(x))$
$Mcos(ln(x))$	$Axsin(ln(x)) + Bxcos(ln(x))$

Similarly to our constant coefficient second order differential equations, there may be overlap between the solutions of y_c and $F(x)$. In this case, you must multiply your guess by $ln(x)$, not t.