1.2 Matrix arithmetic

Now that we have a general idea of what a matrix is, we can discuss the math that we can do with matrices. The main focus of this course on matrix algebra includes matrix addition & subtraction, scalar multiplication, and matrix multiplication.

Matrix addition & subtraction

In order to add (or equivalently, subtract) two matrices, they must be the same size. This is because when we add or subtract, we do it entry-wise, so we take the corresponding entries in each matrix and add or subtract them accordingly.

In order to add (or equivalently, subtract) two matrices, they must be the same size. This is because we perform the operation entry-wise: for any arbitrary $m \times n$ matrices A and B, where m and $n \in \mathbb{Z}^+$, the matrices A and B must have the same dimensions. When we add two $m \times n$ matrices, A and B, each corresponding entry is added to form the resulting matrix. This means that $(A + B)_{i,j} = a_{i,j} + b_{i,j}$, where $a_{i,j}$ and $b_{i,j}$ are the entries of A and B respectively, and i and j represent the row and column indices. Here is an example of matrix addition for two 2x2 matrices:

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -1 & 10 \end{bmatrix}$$

All of these conventions are the same for matrix subtraction, $(A - B)_{i,j} = a_{i,j} - b_{i,j}$. It is important to note that matrix addition is commutative and the order of the matrices does not matter. Obviously matrix subtraction is not commutative but follows the same conventions as integer subtraction.

Matrix scalar multiplication

We can also multiply a matrix by a scalar number $\in \mathbb{R}$. In this case, each entry of the matrix is integer-multiplied by that scalar. Therefore for any $m \times n$ matrix, A, $c \times A$ results in each entry $a_{i,j}$ becoming $c \times a_{i,j}$.

Here is an example of matrix scalar multiplication for a 3x2 matrix:

$$5 \times \begin{bmatrix} 2 & 3 \\ 5 & 4 \\ 1 & -6 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 25 & 20 \\ 5 & -30 \end{bmatrix}$$

Matrix multiplication

Lastly, we attack the the most complicated arithmetic with matrices — matrix multiplication. Matrix multiplication is **not** commutative, the order in which you do matrix multiplication matters, $A \times B \neq B \times A$.

Further, you cannot multiply any two matrices together. Like we discussed earlier, a matrix is denoted by its rows x columns. You can only multiply two matrices if the number of columns of the first matrix is the same as the number of rows of the second matrix.

For example, you **can** multiply a 3×2 matrix by a 2×4 matrix, which would get a resultant 3×4 matrix, but you **cannot** multiply the other way, a 2×4 matrix by a 3×2 matrix. It does not compute and you would get an error in a calculator.

Now, let's discuss exactly how to do this.

The gist of matrix multiplication is for every entry of the product matrix, it is the dot product of the associated row of the first matrix and the column of the second matrix. For example,
This process is easy to see in two 2×2 matrices, A and B.

$$A \times B = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \times \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}(b_{1,1}) + a_{1,2}(b_{2,1}) & a_{1,1}(b_{1,2}) + a_{1,2}(b_{2,2}) \\ a_{2,1}(b_{1,1}) + a_{2,2}(b_{2,1}) & a_{2,1}(b_{1,2}) + a_{2,2}(b_{2,2}) \end{bmatrix}$$

Let's look at this example below of a 3×2 x 2×4.

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 1 & -1 \\ 4 & 5 & -2 & -3 \end{bmatrix} =$$

$$\begin{bmatrix} 2(2) + 3(4) & 2(3) + 3(5) & 2(1) + 2(-2) & 2(-1) + 3(-3) \\ -1(2) + 4(4) & -1(3) + 4(5) & -1(1) + 4(-2) & -1(-1) + 4(-3) \\ -3(2) + 1(4) & -3(3) + 1(5) & -3(1) + 1(-2) & -3(-1) + 1(-3) \end{bmatrix} = \begin{bmatrix} 16 & 21 & -4 & -11 \\ 14 & 17 & -9 & -11 \\ -2 & -4 & -5 & 0 \end{bmatrix}$$

Since the matrices being multiplied are 3×2 x 2×4, and since the middle numbers match (# columns of the first matrix = # rows of the second matrix), the matrix product does exist. Since this number is 2, we do the dot product of each row of the first matrix, of length 2, dotted with each column of the second matrix, of length 2. Since the first matrix has 3 rows, the resultant matrix has 3 rows. Since the second matrix has 4 columns, the resultant matrix has 4 columns.